

ON THE REGULAR REPRESENTATION OF AN (ESSENTIALLY) FINITE 2-GROUP

JOSEP ELGUETA

ABSTRACT. The regular representation of an essentially finite 2-group \mathbb{G} in the 2-category $\mathbf{2Vect}_k$ of (Kapranov and Voevodsky) 2-vector spaces is defined and cohomology invariants classifying it computed. It is next shown that all hom-categories in $\mathbf{Rep}_{\mathbf{2Vect}_k}(\mathbb{G})$ are 2-vector spaces under quite standard assumptions on the field k , and a formula giving the corresponding “intertwining numbers” is obtained which proves they are symmetric. Finally, it is shown that the forgetful 2-functor $\omega : \mathbf{Rep}_{\mathbf{2Vect}_k}(\mathbb{G}) \rightarrow \mathbf{2Vect}_k$ is representable with the regular representation as representing object. As a consequence we obtain a k -linear equivalence between the 2-vector space $\mathbf{Vect}_k^{\mathbb{G}}$ of functors from the underlying groupoid of \mathbb{G} to \mathbf{Vect}_k , on the one hand, and the k -linear category $\mathcal{E}nd(\omega)$ of pseudonatural endomorphisms of ω , on the other hand. We conclude that $\mathcal{E}nd(\omega)$ is a 2-vector space, and we (partially) describe a basis of it.

1. INTRODUCTION

Representation theory of 2-groups, i.e. of categories with a structure analogous to that of a group, is a quite recent subject. Although the special case of *discrete* 2-groups (2-groups whose underlying category is discrete) was already considered in the 1990’s as (weak) actions of groups on categories (see [6]), the first works concerning general 2-groups appeared as preprints in the current decade ([5], [11], [3], [8]).

But what is a representation of a 2-group? By a representation of a group it is meant its representation as a group of automorphisms of an object in some category, mostly the category \mathbf{Set}_f of finite sets or the category \mathbf{Vect}_k of (finite dimensional) vector spaces over a field k . Similarly, by a representation of a 2-group \mathbb{G} one means its representation as a 2-group of (weak) automorphisms of an object in some 2-category \mathbf{C} . For instance, in a representation of \mathbb{G} in the 2-category \mathbf{Cat} of (small) categories, functors and natural transformations the objects of \mathbb{G} are thought of as self-equivalences of a certain category \mathcal{C} and the morphisms as natural isomorphisms between these self-equivalences. This considerably generalizes, for instance, the theory of representations of (finite) groups as permutations of a (finite) set, recovered as the representations of the associated (finite) discrete 2-group when \mathcal{C} is a (finite) discrete category.

Clearly, the first question one has to face when studying representations of 2-groups is what 2-category we should take as \mathbf{C} . In [8] I considered representations of 2-groups in the 2-category $\mathbf{2Vect}_k$ of Kapranov and Voevodsky 2-vector spaces over a field k . This is a higher dimensional version of the category \mathbf{Vect}_k where the role of the field k is played by the (semiring) category \mathbf{Vect}_k .

The natural question arises whether this is a good choice. The answer obviously depends on what one means by “good”. A reasonable measure of the “goodness” of a representation theory seems to be the amount of information on the 2-group we are able to recover from the corresponding (2-)category of representations. In the case of groups, a representation theory which has proved good, at least for some kinds of groups, is the theory of complex finite dimensional linear representations. Under appropriate assumptions on the group, it can indeed be completely recovered from the corresponding category of such representations. Results of this kind are generically known as *reconstruction theorems*. The first such theorem, going back to the 1930’s, is Pontryagin’s duality theorem on the canonical isomorphism between any locally compact abelian topological group and its topological bidual [19]. In this case, we are able to recover the original group from just the group of isomorphism classes of 1-dimensional representations. Later on, Tannaka and Krein concentrated

on the problem of reconstructing any compact topological group G , not necessarily abelian, from the whole ring of isomorphism classes of finite dimensional linear representations. Stated in a more modern categorical language [20], they proved that the canonical map $\pi : G \rightarrow \text{End}(\omega)$ sending any $g \in G$ to the endomorphism of the forgetful functor $\omega : \mathcal{R}ep_{\mathbf{Vect}_k}(G) \rightarrow \mathbf{Vect}_k$ with components $\pi(g)_{(V,\rho)} = \rho(g)$ defines an isomorphism of topological groups between G and the group $\text{Aut}_{\otimes}(\omega) \subset \text{End}(\omega)$ of *monoidal* automorphisms of ω .

Although $\mathbf{2Vect}_k$ is introduced as a sort of higher dimensional analog of \mathbf{Vect}_k it is pretty clear that for many 2-groups the representation theory in $\mathbf{2Vect}_k$ will have deficiencies. Indeed, it is easy to see [8] that a representation of a 2-group \mathbb{G} in $\mathbf{2Vect}_k$ is given, among other things, by a representation of the group $\pi_0(\mathbb{G})$ of isomorphism classes of objects of \mathbb{G} as automorphisms of a *finite* set. Here we think of $\pi_0(\mathbb{G})$ as a group with the group law induced by the product existing between objects. Hence, for infinite 2-groups, in particular, for ‘Lie 2-groups’ (see [2]) there will be very few such representations and we will not be able to reconstruct the whole 2-group from them.

However, it is plausible that this representation theory is good enough if we restrict to essentially finite 2-groups, i.e. 2-groups whose underlying category has a finite set of isomorphism classes of objects and a finite set of morphisms between any given objects.

This paper is a natural continuation of the research program initiated in [8] with the purpose of investigating the representation theory of 2-groups in this kind of 2-vector spaces. More particularly, it arises as a first step toward the proof of the previous guess. Indeed one of the goals of the program is to prove that any essentially finite 2-group \mathbb{G} can be recovered as the 2-group $\text{Aut}_{\otimes}(\omega)$ of monoidal automorphisms of the forgetful 2-functor $\omega : \mathbf{Rep}_{\mathbf{2Vect}_k}(\mathbb{G}) \rightarrow \mathbf{2Vect}_k$ mapping any representation of \mathbb{G} in $\mathbf{2Vect}_k$ to its underlying 2-vector space. This would translate into the category setting the classical result we have for finite groups and its finite dimensional linear representations.

Indeed, for a finite group G the above mentioned theorem identifying G with the group $\text{Aut}_{\otimes}(\omega)$ can be proved using the regular representation $L(G)$ of G and the fundamental fact that this representation represents the forgetful functor $\omega : \mathcal{R}ep_{\mathbf{Vect}_k}(G) \rightarrow \mathbf{Vect}_k$. By the (enriched version of the) Yoneda lemma it follows that there exists a linear isomorphism $\phi : L(G) \xrightarrow{\cong} \text{End}(\omega)$, which is essentially an extension of the canonical map $\pi : G \rightarrow \text{End}(\omega)$ mentioned before. The point is that $L(G)$ has a structure of a Hopf algebra whose group-like part is, on the one hand, isomorphic to G and, on the other hand, bijectively mapped by ϕ to the subset of monoidal endomorphisms of ω .

With this situation in mind, the main purpose of this work is to introduce an analogue of the regular representation for essentially finite 2-groups \mathbb{G} and to see, using the appropriate 2-categorical version of the Yoneda lemma, that it indeed represents the corresponding forgetful 2-functor ω . For this to make sense, it is first necessary to prove that the 2-category of representations of an essentially finite 2-group in $\mathbf{2Vect}_k$ is ‘closed’ in the sense that all its hom-categories are still 2-vector spaces. This is not true for an arbitrary field k , but we shall prove it under quite standard assumptions on k . This allows us to define a k -linear equivalence of categories $\mathbf{Vect}_k^{\mathcal{G}} \xrightarrow{\cong} \mathcal{E}nd(\omega)$ analogous to the above k -linear isomorphism $\phi : L(G) \xrightarrow{\cong} \text{End}(\omega)$, where $\mathbf{Vect}_k^{\mathcal{G}}$ denotes the k -linear category of all \mathbf{Vect}_k -valued functors on the underlying groupoid \mathcal{G} of \mathbb{G} . In a future paper it is intended to prove that both categories actually admit a natural structure of a *Hopf 2-algebra* (higher dimensional analog of a Hopf algebra) and that this equivalence is in fact as Hopf 2-algebras, providing again an analog in our category setting of well known results in the context of groups.

Outline of the paper. The first three sections serve to recall some definitions and known facts needed in the sequel. Specifically, Section 2 contains a quick review on 2-groups, including their description up to the relevant notion of equivalence, and the basic definitions concerning the representation theory of 2-groups. In section 3 we recall the notion of Kapranov and Voevodsky 2-vector space, give some examples (in particular, the 2-vector space underlying the regular representation of an essentially finite 2-group) and discuss the ‘closedness’ of the corresponding 2-category. The classification of the (general linear) 2-group of self-equivalences of an arbitrary 2-vector space is

also recalled here. Finally, in Section 4 we recall from [8] the cohomological description of the representations of a 2-group in $\mathbf{2Vect}_k$.

The core of the paper starts with Section 5, where we define the regular representation of an essentially finite 2-group and explicitly compute a set of data which classifies it up to equivalence (Proposition 15).

In Section 6 it is shown that, under appropriate assumptions, the 2-category of representations of an essentially finite 2-group \mathbb{G} in $\mathbf{2Vect}_k$ is indeed ‘closed’ in the above sense. The main result is Theorem 21, where it is shown that all hom-categories are equivalent to a product of categories of projective representations (with given central charges) of a certain family of subgroups of $\pi_0(\mathbb{G})$. We also obtain a formula for computing the ranks of the 2-vector spaces one obtains as categories of intertwiners, analogous to the so called intertwining numbers, and we show that they are symmetric.

Finally, in Section 7 we prove that the regular representation of an essentially finite 2-group represents the forgetful 2-functor by identifying a universal object in the underlying 2-vector space of the representation (Theorem 30). This allows us to obtain the above mentioned k -linear equivalence between this 2-vector space and the category $\mathcal{E}nd(\omega)$ of (weak) endomorphisms of the forgetful 2-functor ω , and to identify a ‘basis’ of $\mathcal{E}nd(\omega)$. Since any k -linear functor on $\mathcal{E}nd(\omega)$ is determined, up to isomorphism, but the image of a basis, having available a basis may be useful in defining more structure on $\mathcal{E}nd(\omega)$, such as a product or a coproduct. These are expected to play an important role in the proof of the above mentioned reconstruction of \mathbb{G} as a 2-group of symmetries of ω .

Notation and terminology. All over the paper k denotes a fixed field and $k^* = k \setminus \{0\}$. When we write *2-something* we always mean the strict version. Sometimes, this is emphasized by writing explicitly the word *strict*. The only exception to this rule is when *something* = *group*, in which case we always mean the weak version in general. Strict 2-groups are named so. Vertical and horizontal compositions of natural transformations and more generally, of 2-morphisms τ, σ in any 2-category are respectively denoted by $\tau \cdot \sigma$ and $\tau \circ \sigma$. For any set X (resp. category \mathcal{C}), $X[0]$ (resp. $\mathcal{C}[0]$) denotes the corresponding discrete category with only identity arrows (resp. locally discrete 2-category with only identity 2-arrows). For any monoid M (resp. monoidal category \mathcal{M}), $M[1]$ (resp. $\mathcal{M}[1]$) denotes the corresponding one-object category (resp. one-object 2-category). For any natural number $n \geq 1$, $[n]$ denotes the set $\{1, \dots, n\}$. \mathbf{Vect}_k denotes the category of finite dimensional vector spaces over k .

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2. REVIEW ON 2-GROUPS AND THEIR 2-CATEGORIES OF REPRESENTATIONS

We assume the reader is familiar with the basic notions on bicategories and in particular, with their one-object versions, the monoidal categories. See for instance [4] or the short account [16].

2.1. Quick review on 2-groups. By a *2-group* or *categorical group* it is meant a monoidal groupoid $\mathbb{G} = (\mathcal{G}, \otimes, I, a, l, r)$ such that each object A has a weak inverse, i.e. an object A^* such that $A \otimes A^* \cong I \cong A^* \otimes A$. When the monoidal groupoid is strict (the associator a and the left and right unit constraints l, r are identities) and all inverses A^* are strict ($A \otimes A^* = I = A^* \otimes A$) the 2-group is called *strict*.

The simplest examples are groups G thought of as discrete categories $G[0]$, and abelian groups A thought of as one-object 2-groups $A[1]$. In both cases, the tensor product is given by the group law. More generally, for any G -module A we have the so called *split 2-group* $A[1] \rtimes G[0]$. Its set of objects is G , its set of morphisms is $A \times G$, with a pair (a, g) being an automorphism of g , and

composition and tensor product are given by

$$\begin{aligned}(a', g) \circ (a, g) &= (a' + a, g), \\ g \otimes g' &= gg', \\ (a, g) \otimes (a', g') &= (a + g \cdot a', gg').\end{aligned}$$

This is just a special case of the general notion of semidirect product for 2-groups, in this case between $G[0]$ and $A[1]$ (see [12]).

In general, 2-groups arise as symmetries of objects in a 2-category. Thus for any 2-category \mathbf{C} and any object X of \mathbf{C} the groupoid $\mathcal{E}q(X)$ of self-equivalences of X and 2-isomorphisms between these has a canonical structure of a 2-group with the tensor product given by composition of self-equivalences and horizontal composition of 2-morphisms. We shall denote by $\mathbb{E}q(X)$ the 2-group so defined. Notice that it is strict as a monoidal groupoid because \mathbf{C} is assumed to be strict. However, $\mathbb{E}q(X)$ is a non-strict 2-group in general because there may exist objects having no strict inverse (not all self-equivalences of X will be isomorphisms).

As expected, 2-groups are the objects of a 2-category **2Grp** whose 1-morphisms are monoidal functors between the corresponding monoidal groupoids. Hence these are given by pairs $\mathbb{F} = (F, \mu)$ with $F : \mathcal{G} \rightarrow \mathcal{G}'$ a functor and μ a collection of natural isomorphisms $\mu_{A,B} : F(A \otimes B) \xrightarrow{\cong} F(A) \otimes' F(B)$ indexed by pairs of objects of \mathcal{G} and satisfying suitable coherence conditions. As it concerns 2-morphisms, they are given by the so called monoidal natural transformations between these monoidal functors. See [17] for the precise definitions.

A basic result about 2-groups, due to Sinh [21], says that any 2-group \mathbb{G} is equivalent to a sort of “twisted” version of a split 2-group $A[1] \rtimes G[0]$ for some G -module A . More precisely, let $\pi_0(\mathbb{G})$ be the group of isomorphism classes of objects in \mathbb{G} with the product induced by the tensor product, and let $\pi_1(\mathbb{G})$ be the abelian group $\text{Aut}(I)$ of automorphisms of the unit object of \mathbb{G} . This indeed is an abelian group and it has a canonical $\pi_0(\mathbb{G})$ -module structure. Then Sinh’s classification theorem says that \mathbb{G} is equivalent to the semidirect product $\pi_1(\mathbb{G})[1] \rtimes \pi_0(\mathbb{G})[0]$ but equipped with a non-trivial associator $a_{g,g',g''} : gg'g'' \rightarrow gg'g''$ given by

$$a_{g,g',g''} = (\alpha(g, g', g''), gg'g''),$$

where $\alpha \in Z^3(\pi_0(\mathbb{G}), \pi_1(\mathbb{G}))$ is a certain 3-cocycle somehow constructed from the associator of \mathbb{G} . We shall denote the 2-group defined in this way by $\pi_1(\mathbb{G})[1] \rtimes_{\alpha} \pi_0(\mathbb{G})[0]$. For more details cf. [21] or the more accessible reference [2]. The groups $\pi_0(\mathbb{G})$ and $\pi_1(\mathbb{G})$ are called the *homotopy groups* of \mathbb{G} and the cohomology class $[\alpha] \in H^3(\pi_0(\mathbb{G}), \pi_1(\mathbb{G}))$ its *Postnikov invariant*. Thus split 2-groups are those whose Postnikov invariant is $[\alpha] = 0$. Any 3-cocycle α in the Postnikov invariant of \mathbb{G} is called a *classifying 3-cocycle* of \mathbb{G} .

In this paper we will mainly concentrate on *essentially finite* 2-groups, by which we mean 2-groups \mathbb{G} both of whose homotopy groups $\pi_0(\mathbb{G})$ and $\pi_1(\mathbb{G})$ are finite.

2.2. Representation bicategories of a 2-group. The category of representations of a group G in a category \mathcal{C} , such as Vect_k , is nothing but the functor category $\text{Fun}(G[1], \mathcal{C})$. Indeed, a functor $F : G[1] \rightarrow \mathcal{C}$ is given by an arbitrary object X of \mathcal{C} and a morphism of groups $\rho : G \rightarrow \text{Aut}_{\mathcal{C}}(X)$, and it is easy to check that morphisms between representations correspond to natural transformation between the respective functors.

By analogy, for any bicategory (resp. 2-category) \mathbf{C} and any 2-group \mathbb{G} the bicategory (resp. 2-category) of representations of \mathbb{G} in \mathbf{C} is defined as the pseudofunctor bicategory (resp. 2-category)

$$\mathbf{Rep}_{\mathbf{C}}(\mathbb{G}) := \mathbf{PsFun}(\mathbb{G}[1], \mathbf{C}).$$

Hence objects are pseudofunctors $\mathbf{F} : \mathbb{G}[1] \rightarrow \mathbf{C}$, 1-morphisms are pseudonatural transformations between these and 2-morphisms are modifications of pseudonatural transformations. When the notions of pseudofunctor and pseudonatural transformation are unpacked we get for the objects and morphisms in $\mathbf{Rep}_{\mathbf{C}}(\mathbb{G})$ the same kind of things that we get for the objects and morphisms in $\text{Rep}_{\mathcal{C}}(G)$. Thus a representation of \mathbb{G} in \mathbf{C} is given by a pair $\mathbf{F} = (X, \mathbb{F})$, with X an object of \mathbf{C} and $\mathbb{F} : \mathbb{G} \rightarrow \mathbb{E}q(X)$ a morphism of 2-groups, and a 1-morphism or *intertwiner* $\xi : (X, \mathbb{F}) \rightarrow (X', \mathbb{F}')$

is given by a pair $\xi = (f, \Phi)$, with $f : X \rightarrow X'$ a 1-morphism in \mathbf{C} and Φ a family of 2-isomorphisms in \mathbf{C}

$$(2.1) \quad \begin{array}{ccc} X & \xrightarrow{F(A)} & X \\ f \downarrow & \Phi(A) \Leftrightarrow & \downarrow f \\ X' & \xrightarrow{F'(A)} & X' \end{array}$$

indexed by the objects A of \mathcal{G} . These 2-isomorphisms come from the weakening of the action preserving condition in the usual notion of intertwiner. They have to be natural in A and to satisfy some coherence conditions.

In our new setting, however, we further have morphisms between intertwiners. More precisely, given intertwiners $(f, \Phi), (\tilde{f}, \tilde{\Phi}) : (X, \mathbb{F}) \rightarrow (X', \mathbb{F}')$ a 2-morphism or *2-intertwiner* between them is just a 2-morphism $\tau : f \Rightarrow \tilde{f}$ in \mathbf{C} satisfying a naturality condition which involves the 2-cells $\Phi(A)$ and $\tilde{\Phi}(A)$. See [8] for more details.

As in any bicategory, we also have a composition law between intertwiners and two composition laws between 2-intertwiners. Composition between intertwiners is given by the so called “vertical composition” of pseudonatural transformations. More explicitly, if $\xi = (f, \Phi) : (X, \mathbb{F}) \rightarrow (X', \mathbb{F}')$ and $\xi' = (f', \Phi') : (X', \mathbb{F}') \rightarrow (X'', \mathbb{F}'')$ the composite $\xi' \circ \xi$ is described by the pair $(f'f, \Phi' * \Phi)$, with the 2-cell $(\Phi' * \Phi)(A)$ given by the pasting

$$\begin{array}{ccc} X & \xrightarrow{F(A)} & X \\ f \downarrow & \Phi(A) \Leftrightarrow & \downarrow f \\ X' & \xrightarrow{F'(A)} & X' \\ f' \downarrow & \Phi'(A) \Leftrightarrow & \downarrow f' \\ X'' & \xrightarrow{F''(A)} & X'' \end{array}$$

Notice that such a pasting only makes sense when \mathbf{C} is a (strict) 2-category, as it is the case in what follows. Otherwise, we should also include the appropriate associativity constraint 2-cells. As for the two compositions between 2-intertwiners, they are given by the vertical and horizontal composition of the corresponding 2-morphisms in \mathbf{C} .

3. 2-VECTOR SPACES.

3.1. Definition and examples. There exists various notions of 2-vector space. See [1], [9], [14], [18]. In this work we shall use the notion originally introduced by Kapranov and Voevodsky in [14] although in a different guise.

According to Kapranov and Voevodsky, a 2-vector space is just a special kind of what they call a $\mathcal{V}ect_k$ -module category. Roughly, this is a symmetric monoidal category \mathcal{V} , analogous to the abelian group in a vector space, together with a functor $\odot : \mathcal{V}ect_k \times \mathcal{V} \rightarrow \mathcal{V}$, called the *action* of $\mathcal{V}ect_k$ on \mathcal{V} , and suitable natural isomorphisms coming from the weakening of the usual axioms for a multiplication by scalars. Then a 2-vector space is defined as a $\mathcal{V}ect_k$ -module category equivalent to $\mathcal{V}ect_k^n$ for some $n \geq 0$. Here $\mathcal{V}ect_k^n$ is assumed to be equipped with the $\mathcal{V}ect_k$ -action induced by the usual tensor product of vector spaces, i.e.

$$V \odot (V_1, \dots, V_n) = (V \otimes V_1, \dots, V \otimes V_n).$$

Instead of this definition, however, we shall use the following equivalent one. It provides an intrinsic characterization of 2-vector spaces and it is much easier to handle.

Definition 1. *A 2-vector space is a (small) k -additive category \mathcal{V} which admits a finite (possibly empty) basis of absolutely simple objects.*

By a k -additive category it is meant a category enriched over \mathcal{Vect}_k (not just over the category \mathcal{Ab} of abelian groups) and with zero object and all binary biproducts. By an absolutely simple object in such a category it is meant an object having no nonzero subobjects other than itself and such that its vector space of endomorphisms is 1-dimensional. By a finite basis of absolutely simple objects it is meant a finite set of absolutely simple objects $\{V_1, \dots, V_n\}$ such that any nonzero object is isomorphic to a *unique* finite biproduct of them. Stated in this way, the definition is due to Neuchl [18].

Notice that, in contrast to what happens in the case of vector spaces, the basis of absolutely simple objects in a 2-vector space is unique (up to isomorphism, of course). This has important consequences as it concerns the representation theory of 2-groups on these 2-vector spaces.

It readily follows from the above definition that the cartesian product $\mathcal{V} \times \mathcal{V}'$ of two 2-vector spaces $\mathcal{V}, \mathcal{V}'$ is a new 2-vector space. A basis of absolutely simple objects is

$$\{(V_1, 0'), \dots, (V_n, 0'), (0, V'_1), \dots, (0, V'_{n'})\},$$

where $\{V_1, \dots, V_n\}$ and $\{V'_1, \dots, V'_{n'}\}$ are bases of \mathcal{V} and \mathcal{V}' , respectively.

Example 2. The standard examples of 2-vector spaces are the product categories \mathcal{Vect}_k^n for any $n \geq 0$. A basis of absolutely simple objects is given by the objects $\{(0, \dots, \overset{i}{k}, \dots, 0), i \in [n]\}$. Any 2-vector space \mathcal{V} is actually equivalent to \mathcal{Vect}_k^n for some $n \geq 0$, called the *rank* of \mathcal{V} .

Example 3. Let G be a finite group and k an algebraically closed field whose characteristic is zero or prime to the order of G . Then the category $\mathcal{Rep}_{\mathcal{Vect}_k}(G)$ of finite dimensional k -linear representations of G is a 2-vector space of rank equal to the number of conjugacy classes of G . A basis of absolutely simple objects is given by any set of representatives of the equivalence classes of irreducible representations. This example generalizes to the case of projective representations with a given (arbitrary) central charge and more generally, to finite dimensional modules over an arbitrary semisimple k -algebra (see § 6.5 below).

Example 4. For any essentially finite 2-group \mathbb{G} the category $\mathcal{Vect}_k^{\mathbb{G}}$ of all functors $F : \mathcal{G} \rightarrow \mathcal{Vect}_k$ and natural transformations between them is a 2-vector space of rank

$$\text{rank}(\mathcal{Vect}_k^{\mathbb{G}}) = |\pi_0(\mathbb{G})| |\pi_1(\mathbb{G})|.$$

Indeed, for any 2-group, essentially finite or not, it always happens that the automorphism group of any object A of \mathcal{G} is isomorphic to $\pi_1(\mathbb{G})$, even when the underlying groupoid \mathcal{G} is non-connected. Thus we have an equivalence of categories

$$(3.1) \quad \mathcal{G} \simeq \coprod_{g \in \pi_0(\mathbb{G})} \pi_1(\mathbb{G})[1],$$

and therefore

$$\mathcal{Vect}_k^{\mathbb{G}} \simeq \mathcal{Vect}_k^{\coprod_{g \in \pi_0(\mathbb{G})} \pi_1(\mathbb{G})[1]} \cong \prod_{g \in \pi_0(\mathbb{G})} \mathcal{Vect}_k^{\pi_1(\mathbb{G})[1]} = \prod_{g \in \pi_0(\mathbb{G})} \mathcal{Rep}_{\mathcal{Vect}_k}(\pi_1(\mathbb{G})).$$

The claim follows now from the previous example and the fact that $\pi_1(\mathbb{G})$ is a finite abelian group. In particular, let $\pi_1(\mathbb{G})^*$ be the dual group of $\pi_1(\mathbb{G})$, i.e. the group of all group morphisms $\chi : \pi_1(\mathbb{G}) \rightarrow k^*$. Then a basis of absolutely simple objects is given by the family of functors

$$\{\eta_{\chi, g} : \mathcal{G} \rightarrow \mathcal{Vect}_k, \chi \in \pi_1(\mathbb{G})^*, g \in \pi_0(\mathbb{G})\}$$

defined on objects A by

$$\eta_{\chi, g}(A) := \begin{cases} k, & \text{if } A \in g \\ 0, & \text{otherwise,} \end{cases}$$

and on morphisms $\varphi : A \rightarrow B$, with $A, B \in g$, by

$$\eta_{\chi, g}(\varphi) = \chi(h_{A, B}^{-1}(\varphi)) \text{ id}_k.$$

Here $h_{A, B} : \pi_1(\mathbb{G}) \rightarrow \text{Hom}(A, B)$ for $A \cong B$ denote isomorphisms we necessarily have to fix if we want to specify any particular set of basic functors $\eta_{\chi, g}$. Thus although for any object A there is a canonical¹ isomorphism $\gamma_A : \pi_1(\mathbb{G}) \xrightarrow{\cong} \text{Aut}(A)$ there is no canonical choice for the isomorphisms $\pi_1(\mathbb{G}) \cong \text{Hom}(A, B)$ when $A \cong B$ but $A \neq B$. Specifying such isomorphisms is best done by choosing representatives A_1, \dots, A_k in each isomorphism class $g \in \pi_0(\mathbb{G})$, with A_1 equal to the unit object I of \mathbb{G} , together with isomorphisms $\iota_A : A \rightarrow A_i$ between each object A and its representative A_i . Making these choices actually amounts to fixing an equivalence of categories as in (3.1). Then an isomorphism $h_{A, B}$ is given by $h_{A, B}^{-1}(\varphi) = \gamma_{A_i}^{-1}(\iota_B \varphi \iota_A^{-1})$. Different choices lead to different isomorphisms $h_{A, B}$ and hence, to different (but isomorphic) basic functors $\eta_{\chi, g}$. To get the decomposition of an arbitrary functor $\eta : \mathcal{G} \rightarrow \text{Vect}_k$ as a biproduct of the $\eta_{\chi, g}$ we just need to take the restriction of η to the various subgroupoids $\text{Aut}(A_i)[1]$ and decompose them as a direct sum of irreps.

Let \mathbf{Cat}_k be the 2-category of all (small) k -linear categories, k -linear functors and natural transformations. Then we denote by $\mathbf{2Vect}_k$ its full sub-2-category with objects all 2-vector spaces. Observe that we still have a third 2-category in between them. Namely, the full sub-2-category \mathbf{AdCat}_k of \mathbf{Cat}_k with objects all k -additive categories.

For any two objects $\mathcal{V}, \mathcal{V}'$ in $\mathbf{2Vect}_k$ the corresponding hom-category is denoted by $\mathcal{H}om_k(\mathcal{V}, \mathcal{V}')$ instead of $\mathcal{H}om_{\mathbf{2Vect}_k}(\mathcal{V}, \mathcal{V}')$. Observe that $\mathbf{2Vect}_k$ is a *replete* sub-2-category of \mathbf{Cat}_k in the sense that any object of \mathbf{Cat}_k equivalent (in \mathbf{Cat}_k) to a 2-vector space is itself a 2-vector space. In fact, any k -linear equivalence between 2-vector spaces maps a basis of absolutely simple objects to a basis of the same kind.

3.2. Hom-categories in $\mathbf{2Vect}_k$. As in the vector spaces setting, all hom-categories in $\mathbf{2Vect}_k$ are themselves 2-vector spaces. Because of its importance we include here the proof of this elementary but fundamental fact.

Proposition 5. *Let $\mathcal{V}, \mathcal{V}'$ be any 2-vector spaces of ranks n, n' respectively. Then $\mathcal{H}om_k(\mathcal{V}, \mathcal{V}')$ is a 2-vector space of rank nn' .*

Proof. The category $\mathcal{H}om_k(\mathcal{V}, \mathcal{V}')$ has an obvious k -additive structure, with the ‘zero functor’ mapping all objects of \mathcal{V} to any fixed zero object of \mathcal{V}' as a zero object of $\mathcal{H}om_k(\mathcal{V}, \mathcal{V}')$, and with the biproduct $H \oplus \tilde{H}$ of any pair $H, \tilde{H} : \mathcal{V} \rightarrow \mathcal{V}'$ of k -linear functors computed pointwise.

The existence of a finite basis follows from the general fact that, up to isomorphism, a k -linear functor $H : \mathcal{V} \rightarrow \mathcal{V}'$ is completely given by the corresponding *matrix of ranks* $R = (r_{i' i}) \in \text{Mat}_{n' \times n}(\mathbb{N})$. By definition, it is the matrix whose entries are uniquely determined by the condition

$$H(V_i) \cong \bigoplus_{i'=1}^{n'} r_{i' i} V_{i'}, \quad i \in [n],$$

where $\{V_1, \dots, V_n\}$ and $\{V'_1, \dots, V'_{n'}\}$ are bases of absolutely simple objects of \mathcal{V} and \mathcal{V}' , respectively. The matrix of ranks of the biproduct of two functors corresponds to taking the sum of the respective matrices of ranks. Hence a basis of $\mathcal{H}om_k(\mathcal{V}, \mathcal{V}')$ is given by any representative set of k -linear functors

$$\{H_{i' i}, (i', i) \in [n'] \times [n]\}$$

whose isomorphism classes are described by the unit matrices (matrices having a unique nonzero entry equal to 1).

¹Actually we have two such canonical isomorphisms, corresponding to the two canonical morphisms $\text{End}(I) \rightarrow \text{End}(X)$ existing for any object X in any monoidal category \mathcal{C} . In case \mathcal{C} is a 2-group these morphisms are isomorphisms; cf. [20], § 1.3.3.3

Once we have fixed biproduct functors in \mathcal{V} and \mathcal{V}' , it is easy to see that any morphism $\tau : H \Rightarrow \tilde{H}$ in $\mathcal{H}om_k(\mathcal{V}, \mathcal{V}')$ is completely given by its ‘basic components’, i.e. the components

$$\tau_{V_i} : \bigoplus_{i'=1}^{n'} r_{i'i} V_{i'}' \rightarrow \bigoplus_{i'=1}^{n'} \tilde{r}_{i'i} V_{i'}', \quad i = 1, \dots, n$$

for a basis $\{V_1, \dots, V_n\}$ of \mathcal{V} . Moreover, each of these components τ_{V_i} is in turn described by a collection of n' arbitrary matrices $M_{i'i} \in \text{Mat}_{\tilde{r}_{i'i} \times r_{i'i}}(k)$, $i' = 1, \dots, n'$, giving the morphism between the homologous “isotypic” pieces

$$M_{i'i} : V_{i'}' \oplus \dots \oplus V_{i'}' \xrightarrow{r_{i'i}} V_{i'}' \oplus \dots \oplus V_{i'}' \xrightarrow{\tilde{r}_{i'i}} V_{i'}' \oplus \dots \oplus V_{i'}', \quad i' = 1, \dots, n'$$

(if both $r_{i'i}, \tilde{r}_{i'i} \neq 0$; otherwise, they are empty matrices). See [7] for more details. In particular, any natural endomorphism of a basic functor $H_{i',i}$ is completely given by an (arbitrary) scalar $\lambda \in k$, and this shows they are indeed absolutely simple. \square

3.3. General linear 2-groups. For any 2-vector space \mathcal{V} we shall denote by $\mathbb{GL}(\mathcal{V})$ the corresponding 2-group of (k -linear) self-equivalences, and by $\mathcal{GL}(\mathcal{V})$ the underlying groupoid. These 2-groups $\mathbb{GL}(\mathcal{V})$ should be thought of as analogs in our category setting of the usual general linear groups, and they will be called *general linear 2-groups*. The underlying monoidal groupoids are always strict because $\mathbf{2Vect}_k$ is a strict 2-category. However, they are non-strict 2-groups in general because there may exist no strict inverses for objects. If n is the rank of \mathcal{V} , it may be shown that $\mathbb{GL}(\mathcal{V})$ is a split 2-group with homotopy groups

$$\begin{aligned} \pi_0(\mathbb{GL}(\mathcal{V})) &\cong S_n, \\ \pi_1(\mathbb{GL}(\mathcal{V})) &\cong (k^*)^n \end{aligned}$$

and with the usual action of S_n on $(k^*)^n$. For the details, see for ex. [9], where these 2-groups are computed for a more general kind of 2-vector spaces including those of Kapranov and Voevodsky.

4. LINEAR REPRESENTATIONS OF A 2-GROUP.

4.1. Description up to equivalence. Let $\mathbf{Rep}_{\mathbf{2Vect}_k}(\mathbb{G})$ be the 2-category of representations of \mathbb{G} in $\mathbf{2Vect}_k$. Thus an object is a pair $\mathbf{F} = (\mathcal{V}, \mathbb{F})$ with \mathcal{V} a 2-vector space and $\mathbb{F} = (F, \mu) : \mathbb{G} \rightarrow \mathbb{GL}(\mathcal{V})$ a morphism of 2-groups. The rank of \mathcal{V} is called the *dimension* of the representation.

As in any 2-category, two objects \mathbf{F} and \mathbf{F}' are said to be *equivalent* when there exists an equivalence between them, i.e. a *weakly* invertible intertwiner between them. In [8] it is shown that the equivalence class of a representation is completely specified by a quadruple (n, ρ, β, c) with

- $n \geq 0$ a natural number,
- $\rho : \pi_0(\mathbb{G}) \rightarrow S_n$ a morphism of groups, where S_n denotes the symmetric group on n elements,
- $\beta : \pi_1(\mathbb{G}) \rightarrow (k^*)^n_\rho$ a morphism of $\pi_0(\mathbb{G})$ -modules such that $[\beta_*(\alpha)] = 0$ (in the cohomology group $H^3(\pi_0(\mathbb{G}), (k^*)^n_\rho)$), and
- $c \in C^2(\pi_0(\mathbb{G}), (k^*)^n_\rho)$ a normalized 2-cochain such that $\partial c = \beta_*(\alpha)$.

Here α is any classifying 3-cocycle of \mathbb{G} , and $(k^*)^n_\rho$ denotes the abelian group of n -tuples of nonzero elements of k with the $\pi_0(\mathbb{G})$ -module structure induced by ρ and the usual action of S_n on $(k^*)^n$

$$g(\lambda_1, \dots, \lambda_n) = (\lambda_{\rho(g^{-1})(1)}, \dots, \lambda_{\rho(g^{-1})(n)}), \quad g \in \pi_0(\mathbb{G}).$$

Notice that this description is neither canonical nor faithful. It is non-canonical because it depends on the specific 3-cocycle α we choose to describe \mathbb{G} up to equivalence. In particular, the 2-cochain c changes with α . But it is also non-faithful because different quadruples, even for a fixed α , can describe the same equivalence class of representations. More precisely, the two quadruples $(n, \rho, \beta, c), (n', \rho', \beta', c')$ specify the same equivalence class of representations if and only if $n = n'$ and there exists $\sigma \in S_n$ such that $\rho' = \sigma \rho \sigma^{-1}$, $\beta' = \sigma \beta$ and $[c'] = [\sigma c]$.

A specific representation $\mathbf{F} = (\mathcal{V}, \mathbb{F})$ whose equivalence class is described by the quadruple (n, ρ, β, c) is the following:

- $\mathcal{V} = \mathcal{Vect}_k^n$.
- $\mathbb{F} = (F, \mu) : \mathbb{G} \rightarrow \mathbb{GL}(\mathcal{Vect}_k^n)$ is the monoidal functor defined as follows:
 - it maps $A \in \text{Obj}\mathcal{G}$ to the permutation functor

$$F(A) \equiv P_{\rho[A]} : \mathcal{Vect}_k^n \rightarrow \mathcal{Vect}_k^n$$

acting on objects (V_1, \dots, V_n) by

$$P_{\rho[A]}(V_1, \dots, V_n) := (V_{\rho[A](1)}, \dots, V_{\rho[A](n)})$$

($[A]$ denotes the isomorphism class of A and $\rho[A]$ its image by ρ);

- it maps a morphism $\varphi : A \rightarrow B$ of \mathcal{G} to the natural automorphism

$$F(\varphi) : P_{\rho[A]} \Rightarrow P_{\rho[B]}$$

(notice that $[B] = [A]$) whose *basic components*² are

$$F(\varphi)_{(0, \dots, k, \dots, 0)}^i := (0, \dots, \beta_{\rho[A](i)}(h_{A,B}^{-1}(\varphi)) \text{id}_K, \dots, 0)$$

(the isomorphisms $h_{A,B}$ are defined in Example 4 above);

- for any objects A, B of \mathcal{G} the natural isomorphism

$$\mu_{A,B} : P_{\rho[A \otimes B]} \Rightarrow P_{\rho[A]} \circ P_{\rho[B]}$$

(actually, an automorphism) giving the monoidal structure is that whose basic components are

$$(\mu_{A,B})_{(0, \dots, k, \dots, 0)}^i := (0, \dots, c_{\rho[A \otimes B](i)}([A], [B]) \text{id}_K, \dots, 0)$$

We shall denote the representation so defined³ by $\mathbf{F}(n, \rho, \beta, c)$. In particular, we see that n gives the dimension of the representation, ρ and β give the action of the corresponding functor $F : \mathcal{G} \rightarrow \mathcal{GL}(\mathcal{V})$ on objects and morphisms, respectively, and c gives the monoidal structure.

The morphism β admits the following alternative description. The left action of $\pi_0(\mathbb{G})$ on $\pi_1(\mathbb{G})$ induces a left action on $\pi_1(\mathbb{G})^*$ given by

$$(g\chi)(u) = \chi(g^{-1}u), \quad g \in \pi_0(\mathbb{G}), \quad \chi \in \pi_1(\mathbb{G})^*, \quad u \in \pi_1(\mathbb{G}).$$

For any natural number $n \geq 1$ and any morphism of groups $\rho : \pi_0(\mathbb{G}) \rightarrow S_n$, let $[n]_\rho$ be the set $[n] \equiv \{1, \dots, n\}$ equipped with the $\pi_0(\mathbb{G})$ -set structure induced by ρ . Then we have the following.

Lemma 6. *For any pair (n, ρ) as above a morphism of $\pi_0(\mathbb{G})$ -modules $\beta : \pi_1(\mathbb{G}) \rightarrow (k^*)_\rho^n$ is the same thing as a $\pi_0(\mathbb{G})$ -equivariant map $\gamma : [n]_\rho \rightarrow \pi_1(\mathbb{G})^*$.*

Proof. From any β as in the statement we define a map γ also as in the statement by $\gamma(i) = \beta_i$, $i = 1, \dots, n$. It is easy to check that this sets a bijection between both types of maps. \square

This is the same kind of things that Crane and Yetter [5] and Baez et al. [10] obtain for the representations of 2-groups in Yetter's measurable categories.

²Once we fix specific direct sum functors in the codomain category \mathcal{Vect}_k^m , any natural transformation $\tau : H \Rightarrow H'$ between k -linear functors $H, H' : \mathcal{Vect}_k^n \rightarrow \mathcal{Vect}_k^m$ is completely determined by the “basic” components $\tau_{(0, \dots, k, \dots, 0)}^i$ for all $i = 1, \dots, n$. This fact was already mentioned before in the proof of Proposition 5. See for ex. [7] for more details.

³Relative to the direct sum functors fixed in each 2-vector space \mathcal{Vect}_k^n , $n \geq 1$.

4.2. Some examples of linear representations.

Example 7. The 1-dimensional trivial representation, denoted by \mathcal{I} , is defined by the pair $(\mathcal{V}, \mathbb{F})$ with $\mathcal{V} = \text{Vect}_k$ and \mathbb{F} equal to the trivial strict morphism of 2-groups. It corresponds to $n = 1$ and β and c the respective constant maps equal to 1.

Example 8. Any $z \in Z^2(\pi_0(\mathbb{G}), k^*)$ defines a 1-dimensional representation of \mathbb{G} where ρ and β are trivial, and cohomologous cocycles define different but equivalent representations. In fact, for discrete 2-groups $G[0]$ we get in this way a canonical bijection between $H^2(G, k^*)$ and the set of equivalence classes of its 1-dimensional representations in $\mathbf{2Vect}_k$.

Example 9. More generally, for any $n \geq 1$ and any $[z] \in H^2(\pi_0(\mathbb{G}), (k^*)^n)$ we have an n -dimensional representation whose corresponding functor $F : \mathcal{G} \rightarrow \mathcal{GL}(\text{Vect}_k^{\mathcal{G}})$ is the trivial one mapping any object to the identity of $\text{Vect}_k^{\mathcal{G}}$ but equipped with a non-trivial monoidal structure. These are called *cocyclic representations* of \mathbb{G} .

Example 10. Any permutation representation $\rho : \pi_0(\mathbb{G}) \rightarrow S_n$ induces an n -dimensional representation of \mathbb{G} whose corresponding functor $F : \mathcal{G} \rightarrow \mathcal{GL}(\text{Vect}_k^{\mathcal{G}})$ just permutes the objects according to ρ . These are called *permutation representations* of \mathbb{G} . Equivalent permutation representations ρ of $\pi_0(\mathbb{G})$ give rise to equivalent permutation representations of \mathbb{G} . In this way, the theory of permutation representations of $\pi_0(\mathbb{G})$ embeds into the theory of representations of \mathbb{G} in $\mathbf{2Vect}_k$ (for a more precise statement, see Theorem 5.13 in [8]).

Clearly, a generic linear representation of \mathbb{G} is a sort of mixture of a cocyclic and a permutation representation.

5. THE REGULAR REPRESENTATION OF AN ESSENTIALLY FINITE 2-GROUP.

Recall that the *regular representation* of a group G is the permutation representation of G induced by the left action of G on itself by left translations. Equivalently, it is the representation defined by the vector space $L(G)$ of all functions $f : G \rightarrow k$ with (left) G -action given by $(gf)(h) = f(hg)$. In this section we describe an analog of this representation for essentially finite 2-groups and a quadruple (n, ρ, β, c) which classifies it up to equivalence.

5.1. Definition of the regular representation. Let $\mathbb{G} = (\mathcal{G}, \otimes, I, a, l, r)$ be an essentially finite 2-group. A canonical representation $\mathbf{R} = (\mathcal{V}_{\mathbf{R}}, \mathbb{F}_{\mathbf{R}})$ of \mathbb{G} can be obtained as follows. Take as $\mathcal{V}_{\mathbf{R}}$ the 2-vector space $\text{Vect}_k^{\mathcal{G}}$ (cf. Example 4), and as $F_{\mathbf{R}} : \mathcal{G} \rightarrow \mathcal{GL}(\text{Vect}_k^{\mathcal{G}})$ the functor which maps $A \in \text{Obj}\mathcal{G}$ to the k -linear self-equivalence $F_{\mathbf{R}}(A) : \text{Vect}_k^{\mathcal{G}} \rightarrow \text{Vect}_k^{\mathcal{G}}$ acting on objects $\eta : \mathcal{G} \rightarrow \text{Vect}_k$ and morphisms $\tau : \eta \Rightarrow \eta'$ by

$$F_{\mathbf{R}}(A)(\eta) := \eta \circ (- \otimes A), \quad F_{\mathbf{R}}(A)(\tau) := \tau \circ 1_{- \otimes A}.$$

If $\varphi : A \rightarrow B$ is any morphism of \mathcal{G} , $F_{\mathbf{R}}(\varphi)$ is the natural transformation

$$F_{\mathbf{R}}(\varphi) : F_{\mathbf{R}}(A) \Rightarrow F_{\mathbf{R}}(B) : \text{Vect}_k^{\mathcal{G}} \rightarrow \text{Vect}_k^{\mathcal{G}}$$

whose η -component $F_{\mathbf{R}}(\varphi)_{\eta} : \eta \circ (- \otimes A) \Rightarrow \eta \circ (- \otimes B)$ is defined by

$$F_{\mathbf{R}}(\varphi)_{\eta, C} := \eta(\text{id}_C \otimes \varphi), \quad C \in \text{Obj}\mathcal{G}.$$

The point is that the functor $F_{\mathbf{R}}$ so defined has a canonical monoidal structure induced by the associativity constraints in \mathbb{G} . More precisely, we have the following:

Lemma 11. For any $B, C \in \text{Obj}\mathcal{G}$ let $\mu_{B, C} : F_{\mathbf{R}}(B \otimes C) \Rightarrow F_{\mathbf{R}}(B) \circ F_{\mathbf{R}}(C) : \text{Vect}_k^{\mathcal{G}} \rightarrow \text{Vect}_k^{\mathcal{G}}$ be the natural transformation with components $\mu_{B, C; \eta} : \eta \circ (- \otimes (B \otimes C)) \Rightarrow \eta \circ (- \otimes C) \circ (- \otimes B)$ given by

$$\mu_{B, C; \eta} := 1_{\eta} \circ a_{-, B, C}, \quad \eta \in \text{Obj}\text{Vect}_k^{\mathcal{G}},$$

where $a_{-, B, C} : - \otimes (B \otimes C) \Rightarrow (- \otimes C) \circ (- \otimes B)$ is the natural isomorphism defined by the associativity constraints $a_{A, B, C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ of \mathbb{G} . Then $\mu_{B, C}$ is natural in B, C and the collection $\mu = \{\mu_{B, C}\}_{B, C}$ provides $F_{\mathbf{R}}$ with a monoidal structure.

Proof. Note first that the diagram

$$\begin{array}{ccc} \eta \circ (- \otimes (B \otimes C)) & \xrightarrow{1_{\eta \circ a_{-,B,C}}} & \eta \circ (- \otimes C) \circ (- \otimes B) \\ \tau \circ 1_{- \otimes (B \otimes C)} \downarrow & & \downarrow \tau \circ 1_{(- \otimes C)(- \otimes B)} \\ \eta' \circ (- \otimes (B \otimes C)) & \xrightarrow{1_{\eta' \circ a_{-,B,C}}} & \eta' \circ (- \otimes C) \circ (- \otimes B) \end{array}$$

commutes for any $\tau : \eta \Rightarrow \eta'$ by the interchange law, so that $\mu_{B,C;\eta}$ is indeed natural in η . Naturality of $\mu_{B,C}$ in B, C means the commutativity of the diagram

$$\begin{array}{ccc} F_{\mathbf{R}}(B \otimes C) & \xrightarrow{\mu_{B,C}} & F_{\mathbf{R}}(B) \circ F_{\mathbf{R}}(C) \\ F_{\mathbf{R}}(\varphi \otimes \psi) \downarrow & & \downarrow F_{\mathbf{R}}(\varphi) \circ F_{\mathbf{R}}(\psi) \\ F_{\mathbf{R}}(B' \otimes C') & \xrightarrow{\mu_{B',C'}} & F_{\mathbf{R}}(B') \circ F_{\mathbf{R}}(C') \end{array}$$

for all morphisms $\varphi : B \rightarrow B'$, $\psi : C \rightarrow C'$ in \mathcal{G} . Taking components this amounts to the commutativity of the diagrams

$$\begin{array}{ccc} \eta(A \otimes (B \otimes C)) & \xrightarrow{\eta(a_{A,B,C})} & \eta((A \otimes B) \otimes C) \\ \eta(id_A \otimes (\varphi \otimes \psi)) \downarrow & & \downarrow \eta((id_A \otimes \varphi) \otimes \psi) \\ \eta(A \otimes (B' \otimes C')) & \xrightarrow{\eta(a_{A,B',C'})} & \eta((A \otimes B') \otimes C') \end{array}$$

for all $\eta : \mathcal{G} \rightarrow \mathcal{Vect}_k$ and all $A \in \text{Obj}\mathcal{G}$, and these diagrams commute because $a_{A,B,C}$ is natural in B, C . Finally, since the underlying monoidal groupoid of $\mathbb{GL}(\mathcal{Vect}_k^{\mathcal{G}})$ is strict, the coherence condition on μ reduces to the commutativity of the diagram

$$\begin{array}{ccc} F_{\mathbf{R}}(B \otimes (C \otimes D)) & \xrightarrow{F_{\mathbf{R}}(a_{B,C,D})} & F_{\mathbf{R}}((B \otimes C) \otimes D) \\ \mu_{B,C \otimes D} \downarrow & & \downarrow \mu_{B \otimes C, D} \\ F_{\mathbf{R}}(B) \circ F_{\mathbf{R}}(C \otimes D) & \xrightarrow{1_{F_{\mathbf{R}}(B)} \circ \mu_{C,D}} & F_{\mathbf{R}}(B) \circ F_{\mathbf{R}}(C) \circ F_{\mathbf{R}}(D) \\ & & \downarrow \mu_{B,C} \circ 1_{F_{\mathbf{R}}(D)} \\ & & F_{\mathbf{R}}(B \otimes C) \circ F_{\mathbf{R}}(D) \end{array}$$

for any objects B, C, D of \mathcal{G} . Taking again components this amounts to the commutativity of the diagrams

$$\begin{array}{ccc} \eta(A \otimes (B \otimes (C \otimes D))) & \xrightarrow{\eta(id_A \otimes a_{B,C,D})} & \eta(A \otimes ((B \otimes C) \otimes D)) \\ \eta(a_{A,B,C \otimes D}) \downarrow & & \downarrow \eta(a_{A,B \otimes C, D}) \\ \eta((A \otimes B) \otimes (C \otimes D)) & \xrightarrow{\eta(a_{A \otimes B, C, D})} & \eta(((A \otimes B) \otimes C) \otimes D) \\ & & \downarrow \eta(a_{A,B,C} \otimes id_D) \\ & & \eta((A \otimes (B \otimes C)) \otimes D) \end{array}$$

for any $\eta : \mathcal{G} \rightarrow \mathcal{Vect}_k$ and any objects A, B, C, D of \mathcal{G} , and these diagrams commute by the pentagon axiom on the associativity isomorphisms. \square

Definition 12. For any essentially finite 2-group \mathbb{G} the regular representation of \mathbb{G} is the representation \mathbf{R} defined by the pair $(\mathcal{Vect}_k^{\mathcal{G}}, \mathbb{F}_{\mathbf{R}})$ with $\mathbb{F}_{\mathbf{R}} = (F_{\mathbf{R}}, \mu)$ the above morphism of 2-groups.

Example 13. For any finite group $G = \{g_1, \dots, g_n\}$, the regular representation of $G[0]$ is the strict monoidal functor $F_{\mathbf{R}} : G[0] \rightarrow \mathcal{GL}(\text{Vect}_k^G)$ mapping $g \in G$ to the permutation functor $\text{Vect}_k^G \rightarrow \text{Vect}_k^G$ given by $(V_{g_1}, \dots, V_{g_n}) \mapsto (V_{g_1 g}, \dots, V_{g_n g})$.

Example 14. For any finite abelian group A , the regular representation of $A[1]$ is (equivalent to) the strict monoidal functor $R_{\mathbf{R}} : A[1] \rightarrow \mathcal{GL}(\text{Rep}_{\text{Vect}_k}(A))$ mapping the unique object to the identity functor and $a \in A$ to the natural automorphism $F_{\mathbf{R}}(a) : \text{id} \Rightarrow \text{id}$ defined by $F_{\mathbf{R}}(a)_{(V, \rho)} = \rho(a)$ for any representation (V, ρ) of A (observe that $\rho(a)$ indeed is an intertwiner from the representation (V, ρ) to itself because A is abelian). Thus it essentially reduces to the canonical morphism from A into the center $Z(\text{Rep}_{\text{Vect}_k}(A))$ of its category of linear representations.

5.2. Classification. Let $p = |\pi_0(\mathbb{G})|$, $q = |\pi_1(\mathbb{G})|$. We know from Example 4 that \mathbf{R} has dimension $n_{\mathbf{R}} = pq$. In this subsection we describe a particular triple $(\rho_{\mathbf{R}}, \beta_{\mathbf{R}}, c_{\mathbf{R}})$ of the kind described in § 4.1 that classifies \mathbf{R} . Recall that such a triple is unique only “up to conjugation”. In particular, it depends on the choice of a representative of the Postnikov invariant of \mathbb{G} . Let us fix once and for all such a representative $\alpha \in Z^3(\pi_0(\mathbb{G}), \pi_1(\mathbb{G}))$, that we can assume normalized without loss of generality.

Before describing the triple $(\rho_{\mathbf{R}}, \beta_{\mathbf{R}}, c_{\mathbf{R}})$ let us first introduce some notation. Let us denote by $S_p \times \overset{q}{\cdots} \times S_p \hookrightarrow S_{pq}$ the embedding mapping the i^{th} -factor S_p ($i = 1, \dots, q$) to the subgroup of S_{pq} leaving all $j \in [pq]$ invariant except the elements $\{(i-1)p + 1, \dots, ip\}$, which are permuted accordingly. In terms of permutation matrices, this means mapping the permutation matrices (P_1, \dots, P_q) to the block diagonal permutation matrix $P = \text{diag}(P_1, \dots, P_q)$. For any linearly ordered finite group $G = \{g_1 < \dots < g_r\}$ let us further denote by $\kappa : G \rightarrow S_r$ the composite

$$G \hookrightarrow \text{Aut}(G) \xrightarrow{\cong} S_r,$$

where $G \hookrightarrow \text{Aut}(G)$ denotes Cayley’s embedding mapping $g \in G$ to the right translation $g' \mapsto g'g^{-1}$, and \cong stands for the isomorphism of groups induced by the chosen linear order in G .

The starting point to classify \mathbf{R} is the classification of the general linear 2-groups $\mathbb{GL}(\mathcal{V})$ described in § 3. We know that $\pi_0(\text{Vect}_k^G) \cong S_{pq}$, but we need to specify a particular such isomorphism. To do this we choose a linear order in one of the sets of basic functors $\{\eta_{\chi, g}\}$ for Vect_k^G described in Example 4. As explained before, we have various such sets of basic functors and we fix any one of them. Let us further fix linear orders $g_1 < \dots < g_p$ in $\pi_0(\mathbb{G})$ and $\chi_1 < \dots < \chi_q$ in $\pi_1(\mathbb{G})$, and take as linear order in the fixed set of basic functors the lexicographical one, i.e. $\eta_{\chi_1, g_1} < \dots < \eta_{\chi_1, g_p} < \dots < \eta_{\chi_q, g_1} < \dots < \eta_{\chi_q, g_p}$. This way a permutation $\sigma \in S_{pq}$ becomes identified with the isomorphism class of the corresponding permutation functor $\text{Vect}_k^G \rightarrow \text{Vect}_k^G$. Moreover, this automatically specifies a particular isomorphism $\pi_1(\text{Vect}_k^G) \cong (k^*)^{pq}$, namely that sending $u : \text{id}_{\text{Vect}_k^G} \Rightarrow \text{id}_{\text{Vect}_k^G}$ to the corresponding basic components $(u_{\chi_1, g_1}, \dots, u_{\chi_1, g_p}, \dots, u_{\chi_q, g_1}, \dots, u_{\chi_q, g_p})$, which we know are completely given by one non-zero scalar each of them (see proof of Proposition 5). With these choices we have the following.

Proposition 15. *The equivalence class of \mathbf{R} is described by the following triple $(\rho_{\mathbf{R}}, \beta_{\mathbf{R}}, c_{\mathbf{R}})$:*

- (i) $\rho_{\mathbf{R}} : \pi_0(\mathbb{G}) \rightarrow S_{pq}$ is given by the composite

$$\pi_0(\mathbb{G}) \xrightarrow{(\kappa, \dots, \kappa)} S_p \times \overset{q}{\cdots} \times S_p \hookrightarrow S_{pq}.$$

- (ii) $\beta_{\mathbf{R}} : \pi_1(\mathbb{G}) \rightarrow (k^*)_{\rho_{\mathbf{R}}}^{pq}$ is the morphism of $\pi_0(\mathbb{G})$ -modules defined by

$$\beta_{\mathbf{R}}(u) := (\chi_1(g_1 u), \dots, \chi_1(g_p u), \dots, \chi_q(g_1 u), \dots, \chi_q(g_p u)), \quad u \in \pi_1(\mathbb{G}).$$

- (iii) $c_{\mathbf{R}} : \pi_0(\mathbb{G}) \times \pi_0(\mathbb{G}) \rightarrow (k^*)_{\rho_{\mathbf{R}}}^{pq}$ is the normalized 2-cochain defined by

$$c_{\mathbf{R}}(g_i, g_j) := (\chi_1(\alpha(g_1, g_i, g_j)), \dots, \chi_1(\alpha(g_p, g_i, g_j)), \dots, \chi_q(\alpha(g_1, g_i, g_j)), \dots, \chi_q(\alpha(g_p, g_i, g_j)))$$

for all $g_i, g_j \in \pi_0(\mathbb{G})$.

Proof. The proof is an easy but instructive exercise to become familiar with the relationship between morphisms of 2-groups and the associated triples described in § 4.1. For example, let us prove (i). We already know that for any $A \in \text{Obj}\mathcal{G}$ the functor $F_{\mathbf{R}}(A)$ basically amounts to permuting the $\eta_{\chi, g}$, and we want to identify what this permutation is. By definition we have

$$F_{\mathbf{R}}(A)(\eta_{\chi, g'})(B) = \eta_{\chi, g'}(B \otimes A) = \begin{cases} k, & \text{if } B \otimes A \in g' \\ 0, & \text{otherwise.} \end{cases}$$

But $B \otimes A \in g'$ if and only if $B \in g'g^{-1}$, where $g = [A]$. This means that $F_{\mathbf{R}}(A)(\eta_{\chi, g'})$ acts on objects in exactly the same way as $\eta_{\chi, g'g^{-1}}$ and consequently, we have

$$F_{\mathbf{R}}(A)(\eta_{\chi, g'}) \cong \eta_{\chi, g'g^{-1}}.$$

Thus the morphism $\pi_0(\mathbb{G}) \rightarrow \pi_0(\text{Vect}_k^{\mathcal{G}})$ maps g to the isomorphism class of the permutation functor on $\text{Vect}_k^{\mathcal{G}}$ given by $\eta_{\chi, g'} \mapsto \eta_{\chi, g'g^{-1}}$, and under our previous identification $\pi_0(\text{Vect}_k^{\mathcal{G}}) \cong S_{pq}$ this indeed corresponds to the morphism $\rho_{\mathbf{R}}$ defined above. We leave to the reader the proof of (ii) and (iii). She/he can also check that $\beta_{\mathbf{R}}$ indeed is a morphism of $\pi_0(\mathbb{G})$ -modules and that $\partial c_{\mathbf{R}} = \beta_*(\alpha)$. \square

In particular, although strictly speaking the regular representation of a finite group G is something different from the regular representation of the associated discrete 2-group $G[0]$, we see that the former is recovered as the equivalence class of the later. For a finite one-object 2-group $A[1]$ we just get the set of all characters of group A as equivalence class of its regular representation.

Later on we shall use the triple $(\rho_{\mathbf{R}}, \beta_{\mathbf{R}}, c_{\mathbf{R}})$ to get some more information on the regular representation (see Example 29 below).

6. CATEGORIES OF INTERTWINERS.

For any representations \mathbf{F}, \mathbf{F}' let $\text{Hom}_{\mathbb{G}}(\mathbf{F}, \mathbf{F}')$, or just $\text{End}_{\mathbb{G}}(\mathbf{F})$ when $\mathbf{F} = \mathbf{F}'$, be the associated category of intertwiners. It inherits an obvious k -additive structure from the k -additive structures we have in the underlying 2-vector spaces of each representation. In general, however, it is not a 2-vector space because there may be no finite basis of absolutely simple objects. For instance, $\text{End}_{\mathbb{G}}(\mathcal{I})$ is equivalent to the category $\mathcal{R}ep_{\text{Vect}_k}(\pi_0(\mathbb{G}))$ of (finite dimensional) linear representations of $\pi_0(\mathbb{G})$ (see Remark 18 below). However, this is not always a 2-vector space. Even if $\pi_0(\mathbb{G})$ is finite, it may lack to be a 2-vector space unless the field k is algebraically closed and of characteristic zero or prime to the order of $\pi_0(\mathbb{G})$.

At first sight, this is a little bit of a surprise when compared to the corresponding situation for groups (finite or not), where the set of intertwiners between any two finite dimensional linear representations *always* is a finite dimensional vector space. The difference arises from the fact that an intertwiner between representations of a 2-group is not just a k -linear functor between the underlying 2-vector spaces which satisfies some additional conditions. That is to say, $\text{Hom}_{\mathbb{G}}(\mathbf{F}, \mathbf{F}')$ is not a subcategory of $\text{Hom}_k(\mathcal{V}, \mathcal{V}')$. We further have the all-important natural isomorphisms $\Phi(A)$ in (2.1) which come out as additional data we are required to specify to completely define an intertwiner.

The purpose of this section is to prove that the same conditions which ensure $\text{End}_{\mathbb{G}}(\mathcal{I})$ is a 2-vector space (namely, $\pi_0(\mathbb{G})$ finite and k algebraically closed and of characteristic zero or prime to the order of $\pi_0(\mathbb{G})$) are actually enough for the category $\text{Hom}_{\mathbb{G}}(\mathbf{F}, \mathbf{F}')$ to be a 2-vector space for any pair of representations \mathbf{F}, \mathbf{F}' . In doing this we shall be able to describe explicitly a basis of absolutely simple objects for these 2-vector spaces as well as a method for computing the correspondings ranks out of the involved representations. The proof is based on the geometric interpretation of these categories of intertwiners given in [8] and recalled in § 6.3.

All over this section various equivalences of categories are considered whose explicit definitions will be needed in Section 7.

6.1. The k -additive category $\mathcal{H}om_{\mathbb{G}}(\mathbf{F}, \mathbf{F}')$. Let $\mathbf{F} = (\mathcal{V}, \mathbb{F})$, $\mathbf{F}' = (\mathcal{V}', \mathbb{F}')$. Then an object in $\mathcal{H}om_{\mathbb{G}}(\mathbf{F}, \mathbf{F}')$ is given by a pair $\xi = (H, \Phi)$ with $H : \mathcal{V} \rightarrow \mathcal{V}'$ a k -linear functor and $\Phi = \{\Phi(A)\}_{A \in \text{Obj } \mathcal{G}}$ a family of natural isomorphisms of functors

$$(6.1) \quad \begin{array}{ccc} \mathcal{V} & \xrightarrow{H} & \mathcal{V}' \\ F(A) \downarrow & \Phi(A) \Leftrightarrow & \downarrow F'(A) \\ \mathcal{V} & \xrightarrow{H} & \mathcal{V}' \end{array}$$

satisfying appropriate naturality and coherence conditions (see § 2.2). In particular, if R is the matrix of ranks of H (see § 3.2), the existence of such natural isomorphisms implies that R is in the obvious sense invariant under the action of $\pi_0(\mathbb{G})$.

Among the objects in $\mathcal{H}om_{\mathbb{G}}(\mathbf{F}, \mathbf{F}')$ we have the *zero intertwiner*, defined by the pair (H_0, Φ_0) with $H_0 : \mathcal{V} \rightarrow \mathcal{V}'$ “the” zero functor mapping all objects of \mathcal{V} to a given zero object of \mathcal{V}' and with all $\Phi_0(A)$ equal to “identity” natural transformations⁴.

A morphism between two intertwiners (H, Φ) and $(\tilde{H}, \tilde{\Phi})$ is just a natural transformation $\tau : H \Rightarrow \tilde{H}$ satisfying a naturality condition which involves the 2-cells $\Phi(A)$ and $\tilde{\Phi}(A)$. It follows that the zero intertwiner is a zero object of $\mathcal{H}om_{\mathbb{G}}(\mathbf{F}, \mathbf{F}')$ and that $\mathcal{H}om_{\mathbb{G}}(\mathbf{F}, \mathbf{F}')$ inherits a k -linear structure from that existing in \mathcal{V}' and given by

$$(\lambda\tau + \lambda'\tau')_V := \lambda\tau_V + \lambda'\tau'_V, \quad V \in \text{Obj } \mathcal{V},$$

for any $\tau, \tau' : H \Rightarrow \tilde{H} : \mathcal{V} \rightarrow \mathcal{V}'$ and any $\lambda, \lambda' \in k$. In particular, we have a forgetful k -linear functor

$$(6.2) \quad \omega_{\mathbf{F}, \mathbf{F}'} : \mathcal{H}om_{\mathbb{G}}(\mathbf{F}, \mathbf{F}') \rightarrow \mathcal{H}om_k(\mathcal{V}, \mathcal{V}')$$

mapping (H, Φ) to H and equal to the identity on morphisms. Notice, however, that this functor is neither injective nor essentially surjective on objects and that it is a non-full functor.

Biproduts in $\mathcal{H}om_{\mathbb{G}}(\mathbf{F}, \mathbf{F}')$ are obtained from the biproduts in $\mathcal{H}om_k(\mathcal{V}, \mathcal{V}')$. More precisely, for objects (H, Φ) , $(\tilde{H}, \tilde{\Phi})$ their biproduct is the pair $(H \oplus \tilde{H}, \Phi \oplus \tilde{\Phi})$ where $H \oplus \tilde{H}$ is the biproduct in $\mathcal{H}om_k(\mathcal{V}, \mathcal{V}')$ (see proof of Proposition 5) and $(\Phi \oplus \tilde{\Phi})(A)$ is given by the pasting

$$\begin{array}{ccccccc} \mathcal{V} & \xrightarrow{\quad\quad\quad} & \mathcal{V} & \xrightarrow{\quad\quad\quad} & \mathcal{V} & \xrightarrow{H \oplus \tilde{H}} & \mathcal{V}' \\ F(A) \downarrow & \cong & \downarrow h_r & \Phi(A) \oplus \tilde{\Phi}(A) \Leftrightarrow & \downarrow h_l & \cong & \downarrow F'(A) \\ \mathcal{V} & \xrightarrow{H \oplus \tilde{H}} & \mathcal{V}' & \xrightarrow{\quad\quad\quad} & \mathcal{V}' & \xrightarrow{\quad\quad\quad} & \mathcal{V}' \end{array}$$

where $h_r := (H \circ F(A)) \oplus (\tilde{H} \circ F(A))$ and $h_l := (F'(A) \circ H) \oplus (F'(A) \circ \tilde{H})$. This makes sense because composition of k -linear functors is k -bilinear and hence, distributes over biproduts in a canonical way. We leave to the reader checking that the pair $(H \oplus \tilde{H}, \Phi \oplus \tilde{\Phi})$ so defined is indeed a new intertwiner between \mathbf{F} and \mathbf{F}' .

6.2. Notation. If $\mathbf{F}_1 \simeq \mathbf{F}'_1$ and $\mathbf{F}_2 \simeq \mathbf{F}'_2$ we clearly have $\mathcal{H}om_{\mathbb{G}}(\mathbf{F}_1, \mathbf{F}'_1) \simeq \mathcal{H}om_{\mathbb{G}}(\mathbf{F}_2, \mathbf{F}'_2)$. To emphasize this, in the rest of this section we denote the intertwining hom-categories by

$$\mathcal{H}om_{\mathbb{G}}(\mathbf{F}, \mathbf{F}') \equiv \mathcal{H} \left(\begin{array}{c} n, \rho, \beta, c \\ n', \rho', \beta', c' \end{array} \right),$$

or just $\mathcal{H}(n, \rho, \beta, c)$ when both representations are the same (up to equivalence). The reader may think of these categories $\mathcal{H} \left(\begin{array}{c} n, \rho, \beta, c \\ n', \rho', \beta', c' \end{array} \right)$ as the hom-categories between specific representatives we have fixed once and for all for each equivalence class of representations. For instance, those described in § 4.1.

⁴Strictly speaking, the composites $H_0 F(A)$ and $F'(A) H_0$ need not be equal. This is the case if $F'(A)$ maps the given zero object of \mathcal{V}' to another zero object. Anyway, we always have a unique isomorphism between both functors.

6.3. Geometric description of the categories of intertwiners. Let G be any group and X a right G -set. We shall denote by $F(X, k^*)$ the (multiplicative) abelian group of all k^* -valued functions on X . When we speak of 2-cocycles of G with values in $F(X, k^*)$ we always assume $F(X, k^*)$ to be equipped with the G -module structure

$$(g \cdot f)(x) = f(xg), \quad x \in X.$$

Let z be a normalized 2-cocycle of G with values in $F(X, k^*)$ (i.e. a 2-cocycle such that $z(g, e) = z(e, g) = 1$ for any $g \in G$, where 1 denotes the unit of $F(X, k^*)$).

Given (G, X, z) as above, we denote by $\mathcal{Vect}_{G,z}(X)$ the corresponding category of z -projective G -equivariant vector bundles over X . Objects are given by triples (E, p, Θ) with (E, p) a finite rank vector bundle $p : E \rightarrow X$ over X , and $\Theta : E \times G \rightarrow E$ a z -projective right G -action making p a G -equivariant map and whose restriction to fibers is k -linear. Thus if we denote by $\theta(x, g) : E_x \rightarrow E_{xg}$ the k -linear isomorphisms defined by the restriction of Θ to $E_x \times \{g\}$ we have

$$(6.3) \quad \theta(x, gg') = z(g, g')(x) \theta(xg, g') \circ \theta(x, g)$$

$$(6.4) \quad \theta(x, e) = \text{id}_{E_x}$$

for all $g, g' \in G$ and $x \in X$. A morphism $\phi : (E, p, \Theta) \rightarrow (E', p', \Theta')$ between two such triples is an action preserving morphism of vector bundles, hence a family

$$\phi = \{\phi_x : E_x \rightarrow E'_x\}_{x \in X}$$

of k -linear maps such that

$$(6.5) \quad \phi_{xg} \circ \theta(x, g) = \theta'(x, g) \circ \phi_x$$

for all $g \in G$ and $x \in X$. Composition is the obvious one.

Observe that in writing $\mathcal{Vect}_{G,z}(X)$ we do not make explicit the field k . But it is there. Actually, $\mathcal{Vect}_{G,z}(X)$ is a k -additive category. The k -linear structure is the obvious one, the zero vector bundle equipped with its unique z -projective right G -action is a zero object, and $(E, p, \Theta) \oplus (E', p', \Theta')$ is the usual direct sum of vector bundles equipped with the z -projective action

$$(\theta \oplus \theta')(x, g) : E_x \oplus E'_x \rightarrow E_{xg} \oplus E'_{xg}$$

defined by

$$(\theta \oplus \theta')(x, g)(v_x + v'_x) := \theta(x, g)(v_x) + \theta'(x, g)(v'_x), \quad v_x \in E_x, v'_x \in E'_x.$$

As we will see later, it is even a 2-vector space under suitable assumptions.

Let now (n, ρ, β, c) and (n', ρ', β', c') be quadruples of the kind described in § 4.1. The group morphisms ρ and ρ' induce a right action of $\pi_0(\mathbb{G})$ on $X(n', n) := [n'] \times [n]$ given by

$$(i', i) \cdot g = (\rho'(g^{-1})(i'), \rho(g^{-1})(i)), \quad g \in G.$$

Let us denote by $\Lambda(n, \rho, \beta; n', \rho', \beta')$ the corresponding set of *intertwining* $\pi_0(\mathbb{G})$ -orbits, i.e. orbits X_λ such that $\beta_i = \beta'_{i'}$ for all $(i', i) \in X_\lambda$.⁵ Finally, for each intertwining $\pi_0(\mathbb{G})$ -orbit X_λ a normalized 2-cocycle $z_\lambda \in Z^2(\pi_0(\mathbb{G}), F(X_\lambda, k^*))$ is defined by

$$z_\lambda(g_1, g_2)(i', i) = \frac{c'(g_1, g_2)_{i'}}{c(g_1, g_2)_i}$$

for all $g_1, g_2 \in \pi_0(\mathbb{G})$ and $(i', i) \in X_\lambda$. Then we have the following.

Theorem 16 ([8]). *There is an equivalence of k -additive categories*

$$(6.6) \quad \mathcal{H} \left(\begin{matrix} n, \rho, \beta, c \\ n', \rho', \beta', c' \end{matrix} \right) \simeq \prod_{X_\lambda \in \Lambda(n, \rho, \beta; n', \rho', \beta')} \mathcal{Vect}_{\pi_0(\mathbb{G}), z_\lambda}(X_\lambda).$$

⁵Actually, it is easy to see that this condition holds for all points in X_λ if it holds for some (arbitrary) point $(i', i) \in X_\lambda$.

For later use, let us recall from [8] how this equivalence works. Let (H, Φ) be any intertwiner, and let $R = (r_{i'i})$ be the matrix of ranks of the functor H . As mentioned before, R is invariant under the action of $\pi_0(\mathbb{G})$. Hence associated to each orbit $X_\lambda \subset X(n', n)$ we have a well defined nonnegative integer d_λ (the common value of the corresponding entries in R). This gives the rank of the vector bundle $(E(\lambda), p(\lambda))$ over X_λ , and it is easy to see that this rank is necessarily zero unless X_λ is an intertwining orbit. Let us assume without loss of generality that

$$E(\lambda) = \coprod_{x \in X_\lambda} k^{d_\lambda}$$

and that $p(\lambda)$ is the obvious projection. The z_λ -projective action $\Theta(\lambda)$ is now determined by the natural isomorphisms $\Phi(A)$. To be explicit, let us think of the left hand side of (6.6) as the category of intertwiners between the representations $\mathbf{F}(n, \rho, \beta, c), \mathbf{F}'(n', \rho', \beta', c')$ described in § 4.1. Thus the underlying 2-vector spaces $\mathcal{V}, \mathcal{V}'$ are of the form Vect_k^r in both representations, and $F(A)$ and $F'(A)$ are permutation functors $P_{\rho[A]}$ and $P_{\rho'[A]}$, respectively. In this case $\Phi(A)$ is a natural isomorphism

$$\Phi(A) : P_{\rho'[A]}H \Rightarrow HP_{\rho[A]} : \text{Vect}_k^n \rightarrow \text{Vect}_k^{n'}$$

and hence, it is given by an $n' \times n$ matrix whose $(i', i)^{th}$ -entry is itself an invertible matrix

$$\Phi(A)_{i'i} \in \text{GL}(r_{i', \rho[A](i)}, k)$$

with entries in k if $r_{i', \rho[A](i)} \neq 0$ (otherwise, it is the empty matrix; see § 3.2). Then the linear isomorphisms $\theta((i', i), g) : k^{d_\lambda} \rightarrow k^{d_\lambda}$ defining the action $\Theta(\lambda)$ are those which in canonical bases are given by the invertible matrices

$$\theta((i', i), g) \xrightarrow{\text{canonical bases}} (\Phi(A)_{i', \rho(g^{-1})(i)})^{-1} \in \text{GL}(d_\lambda, k), \quad (i', i) \in X_\lambda$$

for any A such that $[A] = g$. Then (6.6) maps the object (H, Φ) to

$$(E(\lambda), p(\lambda), \Theta(\lambda))_{X_\lambda} \in \text{Obj} \left(\prod_{X_\lambda \in \Lambda(n, \rho, \beta; n', \rho', \beta')} \text{Vect}_{\pi_0(\mathbb{G}), z_\lambda}(X_\lambda) \right).$$

The action on morphisms is as follows. Let $\tau : H \Rightarrow \tilde{H} : \mathcal{V} \rightarrow \mathcal{V}'$ be a morphism from (H, Φ) to $(\tilde{H}, \tilde{\Phi})$ for any intertwiners $(H, \Phi), (\tilde{H}, \tilde{\Phi}) : (\mathcal{V}, \mathbb{F}) \rightarrow (\mathcal{V}', \mathbb{F}')$. As pointed out before, τ is completely given by its components $\tau_{V_i} : H(V_i) \rightarrow \tilde{H}(V_i)$ on a basis $\{V_1, \dots, V_n\}$ of \mathcal{V} , and each of these components is in turn described by n' matrices $M_{i'i} \in \text{Mat}_{\tilde{r}_{i'i} \times r_{i'i}}(k)$, $i' = 1, \dots, n'$ (cf. proof of Proposition 5). Then τ gets mapped to the morphism $\phi = (\phi(\lambda))_\lambda$ whose X_λ -component

$$\phi(\lambda) : (E(\lambda), p(\lambda), \Theta(\lambda)) \longrightarrow (\tilde{E}(\lambda), \tilde{p}(\lambda), \tilde{\Theta}(\lambda))$$

is the morphism in $\text{Vect}_{\pi_0(\mathbb{G}), z_\lambda}(X_\lambda)$ given on fibers by these matrices $M_{i'i}$. More precisely, if $(i', i) \in X_\lambda$ the map

$$\phi(\lambda)_{(i', i)} : E(\lambda)_{(i', i)} = k^{d_\lambda} \longrightarrow k^{\tilde{d}_\lambda} = \tilde{E}(\lambda)_{(i', i)}$$

is the k -linear map given in canonical bases by the matrix $M_{i'i}$. The morphism ϕ so defined satisfies (6.5) because of the above mentioned condition on τ involving the 2-cells $\Phi(A)$ and $\tilde{\Phi}(A)$ and ensuring that τ is indeed a 2-intertwiner between (H, Φ) and $(\tilde{H}, \tilde{\Phi})$ (recall that the functor (6.2) is non-full!).

Remark 17. In [8] we proved that this functor is an equivalence of categories. In fact the functor is k -linear and hence, the equivalence is of k -additive categories. Indeed, any k -linear functor between k -additive categories automatically preserves biproducts; see [17], p. 197 where this is shown for the case the commutative ring k is \mathbb{Z} .

Remark 18. If $n = n' = 1$, and $\beta = \beta'$ and $c = c'$ are the trivial maps $\pi_0(\mathbb{G})^3 \rightarrow k^*$ and $\pi_0(\mathbb{G})^2 \rightarrow k^*$, respectively, we have $\mathbf{F}, \mathbf{F}' \simeq \mathcal{I}$. In this case, the right hand side of (6.6) indeed reduces to the category $\mathcal{R}ep_{\text{Vect}_k}(\pi_0(\mathbb{G}))$. In fact, the equivalence is in this case as monoidal categories when $\text{End}_{\mathbb{G}}(\mathcal{I})$ comes equipped with the monoidal structure induced by the composition of endomorphisms and $\mathcal{R}ep_{\text{Vect}_k}(\pi_0(\mathbb{G}))$ with the usual tensor product of representations. This

implies that we shall have no analog of Schur's lemma, at least in its usual version. Indeed, whatever definition we adopt for the irreducible representations in this 2-category setting, the trivial representation \mathcal{I} should be such a representation. But linear representations of groups, in our case of $\pi_0(\mathbb{G})$, have no inverse with respect to tensor product. Therefore \mathcal{I} will be an irreducible representation with lots of non-invertible nonzero endomorphisms.

6.4. The categories $\mathcal{Vect}_{G,z}(X)$ for a transitive G -set X . It readily follows from Theorem 16 and Proposition 5 that $\mathcal{H}\left(\begin{smallmatrix} n, \rho, \beta, c \\ n', \rho', \beta', c' \end{smallmatrix}\right)$ will be a 2-vector space when all k -additive categories $\mathcal{Vect}_{\pi_0(\mathbb{G}), z_\lambda}(X_\lambda)$ are 2-vector spaces. To prove that these categories are indeed 2-vector spaces we shall take advantage of the fact that all $\pi_0(\mathbb{G})$ -sets X_λ are transitive to get a more elementary description of them.

Let us start with the following observation.

Lemma 19. *Let X be a transitive (right) G -set and for any $x \in X$ let $G_x \subset G$ be the stabilizer of x . Then any 2-cocycle $z \in Z^2(G, F(X, k^*))$ gives rise to 2-cocycles $z_x, \hat{z}_x \in Z^2(G_x, k^*)$ defined by*

$$\begin{aligned} z_x(g_1, g_2) &:= z(g_1, g_2)(x) \\ \hat{z}_x(g_1, g_2) &:= z(g_2^{-1}, g_1^{-1})(x) \end{aligned}$$

for any $g_1, g_2 \in G_x$. Here k^* is assumed to be equipped with the trivial G_x -module structure. Furthermore, z_x and \hat{z}_x are normalized when z is normalized.

Proof. An easy computation shows that

$$\begin{aligned} \partial z_x(g_1, g_2, g_3) &= \partial z(g_1, g_2, g_3)(x) \\ \partial \hat{z}_x(g_1, g_2, g_3) &= (\partial z(g_3^{-1}, g_2^{-1}, g_1^{-1}))^{-1}(x) \end{aligned}$$

for all $g_1, g_2, g_3 \in G_x$. □

Recall that for any group H and any normalized 2-cocycle $z \in Z^2(H, k^*)$ a z -projective representation of H (or projective representation with *central charge* z) is a vector space V together with a map $\psi : H \rightarrow GL(V)$ such that $\psi(e) = \text{id}_V$ and

$$\psi(h_1 h_2) = z(h_1, h_2) \psi(h_1) \circ \psi(h_2)$$

for all $h_1, h_2 \in H$. These representations are the objects of a category $\mathcal{Rep}_z(H)$ whose morphisms⁶ are k -linear maps $f : V \rightarrow V'$ such that $f \circ \psi(h) = \psi'(h) \circ f$ for all $h \in H$. In particular, when z is trivial we recover the category of linear representations of H .

Proposition 20. *Let G be a group, X a transitive (right) G -set and $z \in Z^2(G, F(X, k^*))$ a normalized 2-cocycle. Then for any $x_0 \in X$ we have an equivalence of k -additive categories*

$$\mathcal{Vect}_{G,z}(X) \simeq \mathcal{Rep}_{\hat{z}_{x_0}}(G_{x_0}),$$

where $G_{x_0} \subset G$ is the stabilizer (or isotropy subgroup) of x_0 .

Proof. For any object (E, p, Θ) of $\mathcal{Vect}_{G,z}(X)$ let $\psi : G_{x_0} \rightarrow GL(E_{x_0})$ be defined by

$$\psi(g) := \theta(x_0, g^{-1}), \quad g \in G_{x_0}.$$

It readily follows from (6.3) and (6.4) that ψ is a z_{x_0} -projective representation of G_{x_0} . Moreover, from (6.5) it follows that the x_0 -component $\phi_{x_0} : E_{x_0} \rightarrow E'_{x_0}$ of any morphism $\phi : (E, p, \Theta) \rightarrow (E', p', \Theta')$ in $\mathcal{Vect}_{G,z}(X)$ is an intertwiner between the corresponding representations ψ and ψ' . This defines a k -linear functor

$$F : \mathcal{Vect}_{G,z}(X) \rightarrow \mathcal{Rep}_{\hat{z}_{x_0}}(G_{x_0}),$$

⁶Let us remark that there exists a more general notion of morphism between projective representations (with the same or with different central charges) called *projective morphisms*. These are given by a k -linear map $f : V \rightarrow V'$ together with a map $\mu : H \rightarrow k^*$ such that $\mu(1) = 1$ and $f \circ \psi(h) = \mu(h) \psi'(h) \circ f$. When $z = z'$ it follows that μ is a homomorphism. Clearly we have embeddings $\mathcal{Rep}_z(H) \hookrightarrow \mathcal{PRep}_z(H) \hookrightarrow \mathcal{PRep}(H)$, where $\mathcal{PRep}_z(H)$ denotes the category of z -projective representations of H with the projective morphisms, and $\mathcal{PRep}(H)$ the category of z -projective representations of H for arbitrary 2-cocycles z , and projective morphisms between them.

and we claim that this functor is an equivalence of categories.

Indeed, transitivity of X together with (6.5) show that any morphism ϕ in $\mathcal{V}ect_{G,z}(X)$ is uniquely determined by its x_0 -component ϕ_{x_0} and moreover, that any intertwiner $f : E_{x_0} \rightarrow E'_{x_0}$ between ψ and ψ' is the x_0 -component of such a ϕ (i.e. it can be extended to a whole morphism ϕ between (E, p, Θ) and (E', p', Θ')). Hence F is fully faithful.

To prove F is essentially surjective, let $\psi : G_{x_0} \rightarrow \text{GL}(V)$ be any \hat{z}_{x_0} -projective representation. An object of $\mathcal{V}ect_{G,z}(X)$ can be built out of it as follows. Let us fix representatives $\mathcal{R} = \{g_1, \dots, g_r\}$ of the right cosets of G_{x_0}/G , with $g_1 = e$ as representative of G_{x_0} . Set $E := \coprod_{x \in X} V$ and let $p : E \rightarrow X$ be the obvious projection. Because of the transitivity of X , there exist unique $g_i, g_{i'} \in \mathcal{R}$ and $\tilde{g} \in G_{x_0}$ such that

$$(6.7) \quad x = x_0 g_i, \quad g_i g = \tilde{g} g_{i'}.$$

Then for any pair $(x, g) \in X \times G$ let $\theta(x, g) : E_x \rightarrow E_{xg}$ be the k -linear isomorphism defined by

$$(6.8) \quad \theta(x, g) := \frac{z(\tilde{g}, g_{i'})(x_0)}{z(g_i, g)(x_0)} \psi(\tilde{g}^{-1}).$$

Let us see that the pair (E, p) together with these maps indeed define an object of $\mathcal{V}ect_{G,z}(X)$. If $g = e$ we have $\tilde{g} = e$ and $g_{i'} = g_i$. Hence (6.4) holds because z is normalized. To prove (6.3) let $g_j, g_{j'} \in \mathcal{R}$ and $\hat{g} \in G_{x_0}$ be uniquely defined by

$$(6.9) \quad xg = x_0 g_j, \quad g_j g' = \hat{g} g_{j'}.$$

Hence

$$\theta(xg, g') = \frac{z(\hat{g}, g_{j'})(x_0)}{z(g_j, g')(x_0)} \psi(\hat{g}^{-1}).$$

Similarly, let $g_{i''} \in \mathcal{R}$ and $\bar{g} \in G_{x_0}$ be uniquely defined by

$$(6.10) \quad g_i g g' = \bar{g} g_{i''}$$

so that the left hand side of (6.3) is

$$\theta(x, gg') = \frac{z(\bar{g}, g_{i''})(x_0)}{z(g_i, gg')(x_0)} \psi(\bar{g}^{-1}).$$

Thus we have to prove that

$$(6.11) \quad \frac{z(\bar{g}, g_{i''})(x_0)}{z(g_i, gg')(x_0)} \psi(\bar{g}^{-1}) = z(g, g')(x_0 g_i) \frac{z(\hat{g}, g_{j'})(x_0)}{z(g_j, g')(x_0)} \frac{z(\tilde{g}, g_{i'})(x_0)}{z(g_i, g)(x_0)} \psi(\hat{g}^{-1}) \psi(\tilde{g}^{-1})$$

To show this, note first that not all of elements $g_i, g_{i'}, g_{i''}, g_j, g_{j'} \in \mathcal{R}$ are independent, and the same is true for the elements $\tilde{g}, \hat{g}, \bar{g} \in G_{x_0}$. Thus from (6.7) and (6.9) we have

$$x_0 g_j = xg = x_0 g_i g = x_0 \tilde{g} g_{i'} = x_0 g_{i'}$$

so that $g_j = g_{i'}$. Using now (6.10) it follows that

$$\bar{g} g_{i''} = g_i g g' = \tilde{g} g_{i'} g' = \tilde{g} g_j g' = \tilde{g} \hat{g} g_{j'}$$

so that $\bar{g} = \tilde{g} \hat{g}$ and $g_{i''} = g_{j'}$. Moreover we have

$$\psi(\tilde{g}^{-1}) \psi(\hat{g}^{-1}) = \frac{1}{z(\tilde{g}, \hat{g})(x_0)} \psi(\tilde{g}^{-1} \hat{g}^{-1})$$

because ψ is \hat{z}_{x_0} -projective. Putting all these facts together we see that (6.11) reduces to

$$(6.12) \quad z(\tilde{g} \hat{g}, g_{i''}) z(g_{i'}, g') z(g_i, g) z(\tilde{g}, \hat{g})|_{x_0} = (g_i z(g, g')) z(\hat{g}, g_{i''}) z(\tilde{g}, g_{i'}) z(g_i, gg')|_{x_0},$$

where we have used that $z(g, g')(x_0 g_i) = (g_i z(g, g'))(x_0)$. Now by the 2-cocycle condition on z we have

$$\begin{aligned} z(\tilde{g} \hat{g}, g_{i''})(x_0) &= z(\hat{g}, g_{i''}) z(\tilde{g}, \hat{g} g_{i''}) z(\tilde{g}, \hat{g})^{-1}|_{x_0} \\ z(g_i, gg')(x_0) &= (g_i z(g, g'))^{-1} z(g_i g, g') z(g_i, g)|_{x_0}. \end{aligned}$$

In the first equality we have used that $\tilde{g} \in G_{x_0}$ so that $(\tilde{g} z(\hat{g}, g_{i''}))(x_0) = z(\hat{g}, g_{i''})(x_0)$. Putting this into (6.12) and using that $g_i g = \tilde{g} g_{i'}$ and $\hat{g} g_{i''} = g_{i'} g'$ shows that (6.12) holds. To finish the proof

it remains to see that the object (E, p, Θ) of $\mathcal{V}ect_{G,z}(X)$ we have constructed in this way out of ψ indeed gets mapped by the functor F to a \hat{z}_{x_0} -projective representation equivalent to ψ . In fact, it gets mapped to ψ because for any $g \in G_{x_0}$ we have

$$F(E, p, \Theta)(g) = \theta(x_0, g^{-1}) \stackrel{(6.8)}{=} \frac{z(g^{-1}, e)(x_0)}{z(e, g^{-1})(x_0)} \psi(g) = \psi(g).$$

Here we use that the $g_i, g_{i'}, \tilde{g}$ in (6.7) are in this case given by $g_i = g_{i'} = e$ and $\tilde{g} = g^{-1}$ because $x = x_0$ and $g \in G_{x_0}$. \square

Remark 21. We have shown that F is surjective on objects, not just essentially surjective. However, F is not an isomorphism of categories because it is not injective on objects. Indeed, to construct a preimage of ψ we need to choose representatives for the right cosets in G_{x_0}/G , and different choices will give isomorphic, but not equal, objects in $\mathcal{V}ect_{G,z}(X)$ which get mapped to ψ by the functor F . Note also that any pseudoinverse of F will map an intertwiner $f : V \rightarrow V'$ in $\mathcal{R}ep_{z_{x_0}}(G_{x_0})$ to the unique morphism $\phi : \coprod_{x \in X} V \rightarrow \coprod_{x \in X} V'$ whose restriction to the fiber over x_0 is f .

The following is an immediate consequence of the previous result and the obvious fact that $\mathcal{R}ep_z(1) = \mathcal{V}ect_k$.

Corollary 22. *Let X be a G -torsor (i.e. a transitive G -set with trivial stabilizers). Then we have an equivalence of k -additive categories*

$$\mathcal{V}ect_{G,z}(X) \simeq \mathcal{V}ect_k$$

for any normalized 2-cocycle $z \in Z^2(G, F(X, k^*))$.

In particular, we conclude that when X is a G -torsor the isomorphism class of any object (E, p, Θ) of $\mathcal{V}ect_{G,z}(X)$ is completely given by its rank $d \geq 0$. A specific representative in this isomorphism class is the triple $(E(d), p(d), \Theta(d))$ with $E(d) = \coprod_{x \in X} k^d$, $p(d)$ the obvious projection, and $\Theta(d)$ given by

$$(6.13) \quad \theta(d)(x, g) = z(\bar{g}, g)(x_0)^{-1} \text{id}_{k^d}$$

for any $x_0 \in X$ and $\bar{g} \in G$ the unique such that $x = x_0 \bar{g}$; cf. (6.8). Moreover, a k -linear map $f : k^d \rightarrow k^{\tilde{d}}$ corresponds to the morphism $\phi(f) : (E(d), p(d), \Theta(d)) \rightarrow (E(\tilde{d}), p(\tilde{d}), \Theta(\tilde{d}))$ whose components $\phi(f)_x$ are all equal to f . Indeed, as pointed out in the above remark we have $\phi(f)_{x_0} = f$, while the other components follow from (6.13) and (6.5).

6.5. Review on projective representations and modules over arbitrary semisimple algebras. In this subsection we recall a few well known facts from the theory of projective representations of finite groups and more generally, of modules over a semisimple k -algebra. The aim is to see that, under appropriate assumptions on the field k , the corresponding categories (for a given central charge in the case of projective representations) are 2-vector spaces, and to explain how their ranks can be computed. This result generalizes Example 3 in § 3.1 and allows us to prove that all hom-categories in $\mathbf{Rep}_{2\mathcal{V}ect_k}(\mathbb{G})$ indeed are 2-vector spaces under the appropriate assumptions. We refer the reader to [15] for the theory of projective representations of a finite group and to [23] for the general case.

Ordinary linear representations of a finite group G are the same as (left) modules over the group algebra $k[G]$ and moreover, $k[G]$ is a semisimple k -algebra when k is algebraically closed of characteristic zero or prime to the order of G . These are the two basic facts which prove that the category $\mathcal{R}ep_{\mathcal{V}ect_k}(G)$ of Example 3 is a 2-vector space.

More generally, let A be any finite dimensional semisimple k -algebra, with k algebraically closed. Then each finite dimensional A -module decomposes as a finite direct sum of irreducible A -modules, and this decomposition is unique up to isomorphism and permutation of the factors (see Theorem 2.2 in Chapter 2 of [23]). Moreover, irreducible modules are absolutely simple in our sense above, and there are only finitely many isomorphism classes of them (Lemma 2.1 and Corollary 2.15 in Chapter 2 of *loc. cit.*). Briefly, the category $A\text{-Mod}$ of A -modules is a 2-vector space with basis of absolutely

simple objects any set of representatives of the irreducible modules. If $A = k[G]$ the condition on k to be of characteristic zero or prime to the order of G is just the necessary and sufficient condition for $k[G]$ to be semisimple (this is the famous Maschke's theorem; see Theorem 1.14 in Chapter 3 of [23]).

Let us now consider z -projective representations for a given normalized 2-cocycle z . The first remark is that these representations are the same as modules over the *twisted group algebra* $k[G]_z$. This is the k -algebra with the same underlying space as $k[G]$ but with multiplication given by

$$e_g e_{g'} := z(g, g') e_{gg'}, \quad g, g' \in G$$

(cf. Chapter 3, § 2 of [15]). The second remark is that twisted group algebras are also semisimple k -algebras when k is of characteristic zero or prime to the order of G . The proof is essentially the same as for $k[G]$ (see Theorem 2.10 in Chapter 3 of [15]). Therefore, always under the assumption that k is algebraically closed, $\mathcal{R}ep_z(G)$ is a 2-vector space with basis of absolutely simple objects any set of representatives of the irreducible modules.

What about ranks? Let A be an arbitrary finite dimensional semisimple k -algebra, and let $\{M_i, i \in I\}$ be any set of representatives of the isomorphism classes of irreducible A -modules. Then it is shown that $|I| = \dim_k Z(A)$, where $Z(A)$ denotes the center of A , and that $A \cong \bigoplus_{i \in I} n_i M_i$, with the $n_i \geq 0$ such that

$$n_i = \dim_k M_i$$

and

$$\dim_k A = \sum_{i \in I} n_i^2$$

(cf. Corollary 2.24 in Chapter 2 of [23]). In particular, the rank of $A\text{-Mod}$ as a 2-vector space is equal to the dimension over k of the center of A .

This reduces the problem of computing the rank of the 2-vector space $\mathcal{R}ep_z(G)$ to that of computing the dimension over k of the center of $k[G]_z$. If $z = 1$ we recover the usual group algebra $k[G]$, and it is well known that a k -basis of its center is given by the elements $c_i = \sum_{g \in C_i} e_g$, $i = 1, \dots, t$, if C_1, \dots, C_t are the conjugacy classes of G (Lemma 3.2 in Chapter 3 of [23]). This gives the statement in Example 3. The answer for an arbitrary nontrivial normalized 2-cocycle z can be found in [15] (Chapter 3, § 6). If k is of characteristic zero the answer is the following. An element $g \in G$ is called *z -regular* if $e_{g'} e_g = e_g e_{g'}$ for all $g' \in C_g(G)$, the centralizer of g in G . The product here takes place in the twisted group algebra $k[G]_z$. In other words, $g \in G$ is z -regular iff $z(g, g') = z(g', g)$ for all $g' \in C_g(G)$. It is easy to check that if $g \in G$ is z -regular then so is any conjugate of g . Hence it makes sense to speak of the z -regular conjugacy classes of G . Let $C_1, \dots, C_{t'}$ be all the z -regular conjugacy classes of G . Then it is shown that the elements $c_i = \sum_{g \in C_i} e_g$, $i = 1, \dots, t'$, constitute a k -basis of $Z(k[G]_z)$.

Therefore we have the following generalization of Example 3, and a restatement of the above mentioned results from [15].

Proposition 23. *Let k be an algebraically closed field of characteristic zero, G a finite group and $z \in Z^2(G, k^*)$ a normalized 2-cocycle. Then $\mathcal{R}ep_z(G)$ is a 2-vector space of rank the number of z -regular conjugacy classes of G , a basis of absolutely simple objects being given by any set of representatives of the isomorphism classes of irreducible z -projective representations.*

6.6. Main theorem of the section. Next result readily follows now from Theorem 16 and Propositions 20 and 23.

Theorem 24. *Let \mathbb{G} be any 2-group and $(n, \rho, \beta, c), (n', \rho', \beta', c')$ quadruples of the kind described in § 4.1. Then there is a k -linear equivalence of categories*

$$\mathcal{H} \left(\begin{array}{c} n, \rho, \beta, c \\ n', \rho', \beta', c' \end{array} \right) \simeq \prod_{X_\lambda \in \Lambda(n, \rho, \beta; n', \rho', \beta')} \mathcal{R}ep_{z_\lambda}(G_{i'_\lambda i_\lambda}),$$

where (i'_λ, i_λ) is any point of X_λ , $G_{i'_\lambda i_\lambda} \subseteq \pi_0(\mathbb{G})$ the corresponding stabilizer, and $\hat{z}_\lambda \in Z^2(G_{i'_\lambda i_\lambda}, k^*)$ the normalized 2-cocycle defined by

$$\hat{z}_\lambda(g_1, g_2) := \frac{c'(g_2^{-1}, g_1^{-1})_{i'_\lambda}}{c(g_2^{-1}, g_1^{-1})_{i_\lambda}}.$$

Moreover, when \mathbb{G} is essentially finite and k is algebraically closed and of characteristic zero or prime to the order of $\pi_0(\mathbb{G})$ these k -linear categories are 2-vector spaces.

Note that the finiteness of $\pi_0(\mathbb{G})$ ensures that each k -additive category $\mathcal{R}ep_{\hat{z}_\lambda}(G_{i'_\lambda i_\lambda})$ is a 2-vector space, while that of $\pi_1(\mathbb{G})$ ensures that there is a finite number of intertwining orbits in $X(n', n)$ and hence, a finite number of terms in the above product.

The following special case is important for what follows. In particular, it is used in § 7.3 to prove the representability of the forgetful 2-functor by the regular representation (cf. also Corollary 22).

Corollary 25. *Let \mathbb{G} be essentially finite and k algebraically closed and of characteristic zero or prime to the order of $\pi_0(\mathbb{G})$. In the above notations, let us assume that $(n, \rho, \beta, c), (n', \rho', \beta', c')$ are such that all intertwining orbits X_λ are $\pi_0(\mathbb{G})$ -torsors. Then we have a k -linear equivalence of categories*

$$(6.14) \quad \mathcal{H} \left(\begin{array}{c} n, \rho, \beta, c \\ n', \rho', \beta', c' \end{array} \right) \simeq \mathcal{V}ect_k^N,$$

where N is the number of intertwining orbits in $X(n', n)$.

Remark 26. From the whole discussion above it follows that the equivalence (6.14) goes as follows (from right to left). On the one hand, an object $(k^{d_1}, \dots, k^{d_N})$ of $\mathcal{V}ect_k^N$ is mapped to any intertwiner $(H, \Phi) : (\mathcal{V}, \mathbb{F}) \rightarrow (\mathcal{V}', \mathbb{F}')$ whose functor H has a matrix of ranks $R = (r_{i'i})$ given by $r_{i'i} = d_\lambda$ for all $(i', i) \in X_\lambda$ (this completely determines the intertwiner up to isomorphism). In particular, a basis of $\mathcal{H} \left(\begin{array}{c} n, \rho, \beta, c \\ n', \rho', \beta', c' \end{array} \right)$ as a 2-vector space is $\{(H_1, \Phi_1), \dots, (H_N, \Phi_N)\}$ with $(H_\lambda, \Phi_\lambda)$ any intertwiner whose matrix of ranks $R^{(\lambda)}$ is given by

$$(6.15) \quad r_{i'i}^{(\lambda)} = \begin{cases} 1, & \text{if } (i', i) \in X_\lambda \\ 0, & \text{otherwise} \end{cases}$$

On the other hand, a morphism $(f_1, \dots, f_N) : (k^{d_1}, \dots, k^{d_N}) \rightarrow (k^{\tilde{d}_1}, \dots, k^{\tilde{d}_N})$ gets mapped to the unique 2-intertwiner $\tau : H \Rightarrow \tilde{H}$ whose components on the basis ${}^7\{V_1, \dots, V_n\}$ of \mathcal{V} are given by the matrices M_1, \dots, M_N of the linear maps f_1, \dots, f_N in canonical bases. More precisely, if $(i', i) \in X_\lambda$ the restriction of the map $\tau_{V_i} : \oplus_{i'} r_{i'i} V_{i'}' \rightarrow \oplus_{i'} \tilde{r}_{i'i} V_{i'}'$ to its $V_{i'}'$ -isotypic component is that defined by the $(\tilde{d}_\lambda \times d_\lambda)$ -matrix M_λ . Let us emphasize that different matrices M_λ are used to define the same morphism τ_{V_i} , and that there is no obvious general relation between the number of these basic components τ_{V_i} , which is equal to the rank n of \mathcal{V} , and the number of matrices we use to compute them, which is equal to the number N of intertwining orbits. The equivalence so defined is clearly non-canonical. It depends, among other things, on the linear order chosen in the set of intertwining orbits.

6.7. Intertwining numbers. Let us suppose that k is of characteristic zero and that \mathbb{G} is essentially finite. In particular, all hom-categories $\mathcal{H} \left(\begin{array}{c} n, \rho, \beta, c \\ n', \rho', \beta', c' \end{array} \right)$ are 2-vector spaces. Then it follows from the discussion in § 6.5 that the ranks of these 2-vector spaces or *intertwining numbers* can be explicitly computed by the following procedure:

- Find the intertwining $\pi_0(\mathbb{G})$ -orbits X_1, \dots, X_N of $X(n', n)$.
- For each $\lambda = 1, \dots, N$ choose any point $(i'_\lambda, i_\lambda) \in X_\lambda$, determine the corresponding stabilizer $G_{i'_\lambda i_\lambda} \subseteq \pi_0(\mathbb{G})$, and compute the above normalized 2-cocycle $\hat{z}_\lambda : G_\lambda \times G_\lambda \rightarrow k^*$.

⁷The reader may think of $\mathcal{V} = \mathcal{V}ect_k^n$, in which case this basis is given by the objects $(0, \dots, \overset{i}{k}, \dots, 0)$ for all $i = 1, \dots, n$.

- For each $\lambda = 1, \dots, N$ compute the number $r(G_\lambda, \hat{z}_\lambda)$ of \hat{z}_λ -regular conjugacy classes of G_λ .

Then the intertwining number is given by

$$\text{rank } \mathcal{H} \left(\begin{array}{c} n, \rho, \beta, c \\ n', \rho', \beta', c' \end{array} \right) = \sum_{\lambda=1}^N r(G_\lambda, \hat{z}_\lambda).$$

Observe that, proceeding in this way, we only need to take into account that the morphism is from (n, ρ, β, c) to (n', ρ', β', c') , and not in the reverse direction, when computing the 2-cocycles \hat{z}_λ . However, reversing the direction just corresponds to replacing \hat{z}_λ by the inverse 2-cocycle \hat{z}_λ^{-1} . Since the regularity condition of an element $g \in G_\lambda$ is the same either with respect to \hat{z}_λ or with respect to \hat{z}_λ^{-1} , it follows that $r(G_\lambda, \hat{z}_\lambda) = r(G_\lambda, \hat{z}_\lambda^{-1})$. Hence we have the following analog of the well known symmetry property for the intertwining numbers between linear representations of a finite group.

Corollary 27. *Under the above assumptions on \mathbb{G} and k we have*

$$\text{rank } \mathcal{H} \left(\begin{array}{c} n, \rho, \beta, c \\ n', \rho', \beta', c' \end{array} \right) = \text{rank } \mathcal{H} \left(\begin{array}{c} n', \rho', \beta', c' \\ n, \rho, \beta, c \end{array} \right)$$

for any quadruples $(n, \rho, \beta, c), (n', \rho', \beta', c')$.

Example 28. Let us think of the symmetric group S_n as the group of automorphisms of the finite set $[n]$. Then for any morphism of groups $\rho : \pi_0(\mathbb{G}) \rightarrow S_n$ we have

$$\text{rank } \mathcal{H} \left(\begin{array}{c} 1, 1, 1, 1 \\ n, \rho, 1, 1 \end{array} \right) = \sum_{\lambda=1}^s |\{\text{conjugacy classes of } G_s\}|,$$

where G_1, \dots, G_s denote the stabilizers (determined up to conjugation) of the $\pi_0(\mathbb{G})$ -orbits X_1, \dots, X_s of $[n]$. In particular, if we take as ρ Cayley's morphism $\rho_C : \pi_0(\mathbb{G}) \rightarrow S_{|\pi_0(\mathbb{G})|}$ we obtain

$$\text{rank } \mathcal{H} \left(\begin{array}{c} 1, 1, 1, 1 \\ |\pi_0(\mathbb{G})|, \rho_C, 1, 1 \end{array} \right) = |\{\text{conjugacy classes of } \pi_0(\mathbb{G})\}|,$$

in agreement with the fact that we have an equivalence of k -additive categories

$$\mathcal{H} \left(\begin{array}{c} 1, 1, 1, 1 \\ |\pi_0(\mathbb{G})|, \rho_C, 1, 1 \end{array} \right) \simeq \mathcal{R}ep_{\text{vect}_k}(\pi_0(\mathbb{G}))$$

(see Theorem 16). In particular, this is true when $n = 1$ and hence, for the category $\mathcal{E}nd_{\mathbb{G}}(\mathcal{I}) = \mathcal{H} \left(\begin{array}{c} 1, 1, 1, 1 \\ 1, 1, 1, 1 \end{array} \right)$.

Example 29. Let (n, ρ, β, c) be any quadruple of the kind described in § 4.1 and let $(n_{\mathbf{R}}, \rho_{\mathbf{R}}, \beta_{\mathbf{R}}, c_{\mathbf{R}})$ be the quadruple classifying the regular representation of \mathbb{G} (see § 5.2). In particular, we know that $n_{\mathbf{R}} = pq$. Then we have

$$(6.16) \quad \text{rank } \mathcal{H} \left(\begin{array}{c} n_{\mathbf{R}}, \rho_{\mathbf{R}}, \beta_{\mathbf{R}}, c_{\mathbf{R}} \\ n, \rho, \beta, c \end{array} \right) = n.$$

To see this, note first that the stabilizer of any point $(i, k, l) \in [n] \times [q] \times [p] \cong X(n, pq)$ is trivial. Indeed, the point $(k, l) \in [q] \times [p]$ corresponds to the pair $(\chi_k, g_l) \in \pi_1(\mathbb{G})^* \times \pi_0(\mathbb{G})$ (we work with the linear orders we have fixed in § 5.2 for $\pi_0(\mathbb{G})$ and $\pi_1(\mathbb{G})^*$). Hence the action of $g_{l'} \in \pi_0(\mathbb{G})$ on (i, k, l) is

$$(i, k, l) \cdot g_{l'} = (i, \chi_k, g_l) \cdot g_{l'} = (\rho(g_{l'}^{-1})(i), \chi_k, g_l g_{l'})$$

(cf. Proposition 15), and

$$(\rho(g_{l'}^{-1})(i), \chi_k, g_l g_{l'}) \neq (i, \chi_k, g_l) = (i, k, l)$$

unless $g_{l'} = e$. It follows from Corollary 25 that

$$(6.17) \quad \mathcal{H} \left(\begin{array}{c} n_{\mathbf{R}}, \rho_{\mathbf{R}}, \beta_{\mathbf{R}}, c_{\mathbf{R}} \\ n, \rho, \beta, c \end{array} \right) \simeq \mathcal{V}ect_k^N,$$

where $N = N(n_{\mathbf{R}}, \rho_{\mathbf{R}}, \beta_{\mathbf{R}}; n, \rho, \beta)$ is the number of intertwining orbits for the given pair of representations. It remains to see that $N = n$. In fact, we shall determine explicitly the intertwining orbits by identifying a ‘canonical’ representative point in each of them. Let us fix a character $\chi \in \pi_1(\mathbb{G})^*$ and a $\pi_0(\mathbb{G})$ -orbit \mathcal{O} of $[n]_{\rho}$. The subset

$$X_{\mathcal{O}, \chi} := \mathcal{O} \times \{\chi\} \times \pi_0(\mathbb{G}) \subset X(n, pq)$$

is $\pi_0(\mathbb{G})$ -invariant but non-transitive. For example, for any $i \neq i'$ in \mathcal{O} the points (i, χ, e) and (i', χ, e) are not in the same orbit. Actually, the set $\{(i, \chi, e), i \in \mathcal{O}\}$ constitutes a set of representative points for the various orbits of $X_{\mathcal{O}, \chi}$. Indeed, an arbitrary point $(i, \chi, g) \in X_{\mathcal{O}, \chi}$ is in the same orbit as $(\rho(g)(i), \chi, e)$. Therefore, the decomposition of $X_{\mathcal{O}, \chi}$ into orbits looks like

$$X_{\mathcal{O}, \chi} = \coprod_{i \in \mathcal{O}} X_{i, \chi},$$

with $X_{i, \chi} := (i, \chi, e)\pi_0(\mathbb{G})$. In particular, $X_{\mathcal{O}, \chi}$ has $|\mathcal{O}|$ orbits, all of them of cardinal p . Since this is true for each $\chi \in \pi_1(\mathbb{G})^*$ it follows that the decomposition of $X(n, pq)$ into orbits is

$$X(n, pq) = \coprod_{\chi \in \pi_1(\mathbb{G})^*} \coprod_{\mathcal{O} \in \text{Orb}([n]_{\rho})} \coprod_{i \in \mathcal{O}} X_{i, \chi} = \coprod_{\chi \in \pi_1(\mathbb{G})^*} \coprod_{i \in [n]} X_{i, \chi}.$$

However, only n of these qn orbits are intertwining. This is because the (χ, e) -component of $\beta_{\mathbf{R}}$ is $\beta_{\mathbf{R}, (\chi, e)} = \chi$ (see Proposition 15). Thus $(i, \chi, e) \in X_{i, \chi}$ is intertwining if and only if $\beta_i = \chi$. Consequently the set of intertwining orbits is

$$(6.18) \quad \Lambda(n_{\mathbf{R}}, \rho_{\mathbf{R}}, \beta_{\mathbf{R}}; n, \rho, \beta) = \{X_{i, \beta_i}, i \in [n]\} = \{(i, \beta_i, e)\pi_0(\mathbb{G}), i \in [n]\},$$

with $\{(i, \beta_i, e), i = 1, \dots, n\}$ as a set of ‘canonical’ representatives. In particular $N = n$ as claimed.

Once more, this example is nothing but the analog in our setting of a similar result concerning the regular representation of a finite group G . Actually, as in the group setting, this is one of the consequences of the more fundamental fact that for any essentially finite 2-group \mathbb{G} the forgetful 2-functor $\omega : \mathbf{Rep}_{\mathbf{2Vect}_k}(\mathbb{G}) \rightarrow \mathbf{2Vect}_k$ is represented by the regular representation (see Section 7).

6.8. Remarks about k -linear enrichments on 2-categories. By a k -linear (resp. k -additive) 2-category we mean a 2-category \mathbf{C} such that all its hom-categories $\mathcal{H}om_{\mathbf{C}}(X, Y)$ are k -linear (resp. k -additive) and all composition functors

$$\mathcal{H}om_{\mathbf{C}}(X, Y) \times \mathcal{H}om_{\mathbf{C}}(Y, Z) \rightarrow \mathcal{H}om_{\mathbf{C}}(X, Z)$$

are k -bilinear. More particularly, a 2-category will be called a **$\mathbf{2Vect}_k$ -category**⁸ when it is k -additive and all its hom-categories are 2-vector spaces.

The simplest example of a **$\mathbf{2Vect}_k$ -category** is **$\mathbf{2Vect}_k$** itself, which is supposed to be (monoidal) *pseudo-closed* in the sense of [13]. Another example is **$\mathbf{Rep}_{\mathbf{2Vect}_k}(\mathbb{G})$** for \mathbb{G} an essentially finite 2-group and k an algebraically closed field of characteristic zero or prime to the order of $\pi_0(\mathbb{G})$. Indeed, we have shown that all hom-categories in **$\mathbf{Rep}_{\mathbf{2Vect}_k}(\mathbb{G})$** are 2-vector spaces under these assumptions, and the corresponding composition functors are k -bilinear because they are so in **$\mathbf{2Vect}_k$** . For arbitrary \mathbb{G} and k , **$\mathbf{Rep}_{\mathbf{2Vect}_k}(\mathbb{G})$** is just a k -additive 2-category (cf. § 6.1).

Given two k -linear 2-categories \mathbf{C} and \mathbf{D} , a pseudofunctor $\mathbf{F} : \mathbf{C} \rightarrow \mathbf{D}$ is called k -linear when all functors $\mathbf{F}_{X, Y} : \mathcal{H}om_{\mathbf{C}}(X, Y) \rightarrow \mathcal{H}om_{\mathbf{D}}(\mathbf{F}(X), \mathbf{F}(Y))$ are k -linear. An example is the forgetful 2-functor $\omega : \mathbf{Rep}_{\mathbf{2Vect}_k}(\mathbb{G}) \rightarrow \mathbf{2Vect}_k$ mentioned before.

Let us finally remark that for any k -linear 2-category \mathbf{D} the 2-category **$\mathbf{PsFun}(\mathbf{C}, \mathbf{D})$** of pseudofunctors from any other 2-category \mathbf{C} to \mathbf{D} is also k -linear, as the reader may easily check. However, this fails to be true when k -linear is replaced by k -additive. For instance, there may exist no zero object in the hom-categories of **$\mathbf{PsFun}(\mathbf{C}, \mathbf{D})$** even when we have a zero object in each hom-category $\mathcal{H}om_{\mathbf{D}}(X, Y)$. However, we are interested in cases where \mathbf{D} is **$\mathbf{2Vect}_K$** , and **$\mathbf{PsFun}(\mathbf{C}, \mathbf{2Vect}_k)$** is always k -additive. This is because the objects in **$\mathbf{2Vect}_k$** are themselves categories with a zero

⁸It would be more correct to speak of $\mathbf{2Vect}_k$ -categories, where $\mathbf{2Vect}_k$ stands for the underlying category of **$\mathbf{2Vect}_k$** equipped with the monoidal structure defined by the tensor product of k -additive categories (2-vector spaces are indeed stable under this tensor product).

object and binary biproducts, and these can be used to get a zero object and binary biproducts in $\mathbf{PsFun}(\mathbf{C}, \mathbf{2Vect}_k)$. This is in fact how we have seen before that $\mathbf{Rep}_{\mathbf{2Vect}_k}(\mathbb{G})$ is k -additive. The same thing works for the 2-category of pseudofunctors between $\mathbf{Rep}_{\mathbf{2Vect}_k}(\mathbb{G})$ and $\mathbf{2Vect}_k$. In particular, the category $\mathcal{E}nd(\omega)$ of (pseudonatural) endomorphisms of ω is always k -additive. We shall see in the next section that it is even a 2-vector space under suitable assumptions.

7. REPRESENTABILITY OF THE FORGETFUL 2-FUNCTOR ω

In order to prove the representability of ω we shall make use of the appropriate enriched version of the bicategorical Yoneda Lemma. Hence this section starts by recalling this basic result as well as the associated notion of “universal object” for \mathbf{Cat} -valued ($\mathbf{2Vect}_k$ -valued in the enriched case) pseudofunctors. These are analogs of the universal elements of a \mathbf{Set} -valued (resp. \mathbf{Vect}_k -valued) functor. Next it is shown that for essentially finite 2-groups \mathbb{G} and algebraically closed fields k as above there indeed exists a universal object for the forgetful 2-functor $\omega : \mathbf{Rep}_{\mathbf{2Vect}_k}(\mathbb{G}) \rightarrow \mathbf{2Vect}_k$ leading to a representation of it by the regular representation of \mathbb{G} . The section closes with a description of this representation and the induced equivalence between the category $\mathcal{E}nd(\omega)$ of pseudonatural endomorphisms of ω and $\mathbf{Vect}_k^{\mathbb{G}}$. As mentioned in the introduction, this equivalence constitutes a first step toward a Tannaka-Krein type reconstruction of an essentially finite 2-group from its 2-category of representations in 2-vector spaces (and the associated forgetful 2-functor).

7.1. Bicategorical Yoneda Lemma. To my knowledge, this result first appears in its nonenriched version in [22]. It establishes the (natural) equivalence of two categories. More specifically, suppose given a bicategory \mathbf{C} , an object X of \mathbf{C} , and a pseudofunctor $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{Cat}$ with values in the 2-category \mathbf{Cat} of (small) categories, functors and natural transformations. Let us denote by $\mathcal{H}om_{\mathbf{C}}(X, -) : \mathbf{C} \rightarrow \mathbf{Cat}$ the (covariant) hom-pseudofunctor associated to X , and let $\mathcal{P}Nat(\mathcal{H}om_{\mathbf{C}}(X, -), \mathcal{F})$ be the category of all pseudonatural transformations $\mathcal{H}om_{\mathbf{C}}(X, -) \Rightarrow \mathcal{F}$ and modifications between these. Then the bicategorical Yoneda lemma says that there exists an equivalence (not an isomorphism) of categories

$$\mathbf{Yon} : \mathcal{P}Nat(\mathcal{H}om_{\mathbf{C}}(X, -), \mathcal{F}) \xrightarrow{\sim} \mathcal{F}(X)$$

which is given on objects $\xi : \mathcal{H}om_{\mathbf{C}}(X, -) \Rightarrow \mathcal{F}$ by

$$\mathbf{Yon}(\xi) := \xi_X(\mathrm{id}_X),$$

and that this equivalence is natural in X and \mathcal{F} in some suitable sense. Unlike usually for equivalences of categories which are not isomorphisms, \mathbf{Yon} has a canonical pseudoinverse. In fact, although we shall make no use of it, it can be shown that \mathbf{Yon} extends canonically to an adjoint equivalence $(\mathbf{Yon}, \mathbf{Yon}^*, \eta, \epsilon)$ whose unit η is an identity when \mathbf{C} is a 2-category, and whose counit ϵ is an identity when \mathcal{F} is a (strict) 2-functor. The canonical pseudoinverse

$$\mathbf{Yon}^* : \mathcal{F}(X) \rightarrow \mathcal{P}Nat(\mathcal{H}om_{\mathbf{C}}(X, -), \mathcal{F})$$

maps $A \in \mathrm{Obj} \mathcal{F}(X)$ to the pseudonatural transformation $\mathbf{Yon}^*(A) : \mathcal{H}om_{\mathbf{C}}(X, -) \Rightarrow \mathcal{F}$ whose 1-cell components $\mathbf{Yon}^*(A)_Y : \mathcal{H}om_{\mathbf{C}}(X, Y) \rightarrow \mathcal{F}(Y)$ are given on objects $f : X \rightarrow Y$ and morphisms $\tau : f \Rightarrow f'$ by

$$(7.1) \quad \mathbf{Yon}^*(A)_Y(f) := \mathcal{F}(f)(A)$$

$$(7.2) \quad \mathbf{Yon}^*(A)_Y(\tau) := \mathcal{F}(\tau)_A.$$

What we really need is the k -linear version of this lemma. In this version \mathbf{C} is a k -linear 2-category, $\mathcal{F} : \mathbf{C} \rightarrow \mathbf{2Vect}_k$ a k -linear pseudofunctor and

$$(7.3) \quad \mathbf{Yon} : \mathcal{P}Nat(\mathcal{H}om_{\mathbf{C}}(X, -), \mathcal{F}) \xrightarrow{\sim} \mathcal{F}(X)$$

a k -linear equivalence and hence, an equivalence of 2-vector spaces. In particular, the k -linear category $\mathcal{P}Nat(\mathcal{H}om_{\mathbf{C}}(X, -), \mathcal{F})$ is actually a 2-vector space. The pseudoinverse \mathbf{Yon}^* is defined in exactly the same way as before.

7.2. Universal objects of a pseudofunctor. Given a representable functor $F : \mathcal{C} \rightarrow \mathcal{Set}$ with representing object $X \in \text{Obj } \mathcal{C}$, it is well known that the Yoneda bijection

$$\text{Yon} : \text{Nat}(\text{Hom}_{\mathcal{C}}(X, -), F) \xrightarrow{\cong} F(X)$$

restricts to a bijection between representations of F by X (isomorphisms $\text{Hom}_{\mathcal{C}}(X, -) \cong F$) and the so called *universal elements* in $F(X)$. These are elements $x \in F(X)$ such that for any object Y of \mathcal{C} the map $\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow F(Y)$ defined by $f \mapsto F(f)(x)$ is a bijection. The result follows directly from the definition of the bijection to Yon . The analogous result holds for \mathcal{Vect}_k -valued functors and the corresponding Yoneda isomorphisms of vector spaces.

Similarly, given any \mathbf{Cat} -valued or $\mathbf{2Vect}_k$ -valued pseudofunctor \mathcal{F} on a 2-category \mathbf{C} and any object X of \mathbf{C} , by a *universal object* of $\mathcal{F}(X)$ we mean an object $x \in \text{Obj } \mathcal{F}(X)$ such that the pseudonatural transformation $\text{Yon}^*(x)$ is a pseudonatural equivalence. Now it is a general fact that a pseudonatural transformation is an equivalence if and only if all its 1-morphism components are equivalences. Hence $x \in \text{Obj } \mathcal{F}(X)$ is universal if and only if the functors

$$\text{Yon}^*(x)_Y : \text{Hom}_{\mathbf{C}}(X, Y) \rightarrow \mathcal{F}(Y)$$

are equivalences of categories for any $Y \in \text{Obj } \mathbf{C}$.

By the very definition of universal objects, it follows that the Yoneda equivalence (7.3) restricts to a (k -linear) equivalence of categories

$$\text{Yon} : \mathcal{PSeq}(\text{Hom}_{\mathbf{C}}(X, -), \mathcal{F}) \xrightarrow{\cong} \mathcal{F}(X)_u$$

between the full subcategory $\mathcal{PSeq}(\text{Hom}_{\mathbf{C}}(X, -), \mathcal{F})$ with objects all pseudonatural equivalences (i.e. representations of \mathcal{F} by X) and the full subcategory $\mathcal{F}(X)_u$ with objects the universal ones. In particular, the pseudofunctor \mathcal{F} is representable by the object X of \mathbf{C} or equivalently, $\mathcal{PSeq}(\text{Hom}_{\mathbf{C}}(X, -), \mathcal{F})$ is nonempty if and only if there exists such a universal object $x \in \text{Obj } \mathcal{F}(X)$.

7.3. Universal functor $\eta_U : \mathcal{G} \rightarrow \mathcal{Vect}_k$. We are interested in the case where \mathbf{C} is the 2-category $\mathbf{Rep}_{\mathbf{2Vect}_k}(\mathbb{G})$ and \mathcal{F} the forgetful 2-functor $\omega : \mathbf{Rep}_{\mathbf{2Vect}_k}(\mathbb{G}) \rightarrow \mathbf{2Vect}_k$. According to the previous discussion, in order to prove that ω is represented by the regular representation \mathbf{R} it is enough to see that there exists a universal object in the 2-vector space $\omega(\mathbf{R}) = \mathcal{Vect}_k^{\mathcal{G}}$. More precisely, we have to see that there exists a functor

$$\eta_U : \mathcal{G} \rightarrow \mathcal{Vect}_k$$

satisfying the next two conditions:

- (i) For any representation $\mathbf{F} = (\mathcal{V}, \mathbb{F})$ and any $V \in \text{Obj } \mathcal{V}$ there exists an intertwiner $(H, \Phi) : \mathbf{R} \rightarrow \mathbf{F}$ such that $H(\eta_U) \cong V$ (i.e. $\text{Yon}^*(\eta_U)_{\mathbf{F}}$ is essentially surjective; cf. (7.1)).
- (ii) For any representation $\mathbf{F} = (\mathcal{V}, \mathbb{F})$, any intertwiners $(H, \Phi), (\tilde{H}, \tilde{\Phi}) : \mathbf{R} \rightarrow \mathbf{F}$ and any morphism $\phi : H(\eta_U) \rightarrow \tilde{H}(\eta_U)$ in \mathcal{V} there exist a unique 2-intertwiner $\tau : (H, \Phi) \Rightarrow (\tilde{H}, \tilde{\Phi})$ such that $\phi = \tau_{\eta_U}$ (i.e. $\text{Yon}^*(\eta_U)_{\mathbf{F}}$ is fully faithful; cf. (7.2)).

We claim that such a universal functor exists and is given by the direct sum of a few of the basic functors $\{\eta_{\chi, g}, (\chi, g) \in \pi_1(\mathbb{G})^* \times \pi_0(\mathbb{G})\}$ of Example 4. More explicitly, we have the following.

Theorem 30. *Let \mathbb{G} be essentially finite and k algebraically closed and of characteristic zero. Then the pair (\mathbf{R}, η_U) , with \mathbf{R} the regular representation of \mathbb{G} and*

$$\eta_U := \bigoplus_{\chi \in \pi_1(\mathbb{G})^*} \eta_{\chi, e} : \mathcal{G} \rightarrow \mathcal{Vect}_k,$$

is a universal object for the forgetful 2-functor $\omega : \mathbf{Rep}_{\mathbf{2Vect}_k}(\mathbb{G}) \rightarrow \mathbf{2Vect}_k$. In particular, ω is representable with \mathbf{R} as representing object.

Proof. Suppose we are given \mathbf{F} and V as in (i). Let (n, ρ, β, c) be the quadruple classifying \mathbf{F} , and let $\{V_1, \dots, V_n\}$ be a basis of absolutely simple objects of \mathcal{V} . We know that, up to a 2-isomorphism,

any intertwiner $(H, \Phi) : \mathbf{R} \rightarrow \mathbf{F}$ is completely determined by the matrix of ranks of $H : \mathcal{Vect}_k^{\mathcal{G}} \rightarrow \mathcal{V}$ (see (6.17) and Remark 26). Let $R = (r_{j,(\chi,g)})$ be this matrix. Thus we have

$$H(\eta_{\chi,g}) \cong \bigoplus_{j=1}^n r_{j,(\chi,g)} V_j$$

for any $(\chi, g) \in \pi_1(\mathbb{G})^* \times \pi_0(\mathbb{G})$. It follows from the invariance properties of R and the fact that it is supported on the intertwining orbits that this matrix is actually completely given by n integers $d_1, \dots, d_n \geq 0$ giving the values of the nonzero “intertwining entries”. To be precise, we shall assume that d_i gives the value of the entries of R corresponding to the intertwining orbit X_{i,β_i} (see (6.18) for notation). Then let us take as (H, Φ) any intertwiner whose isomorphism class is determined in this way by the unique integers $d_1, \dots, d_n \geq 0$ such that

$$V \cong \bigoplus_{i=1}^n d_i V_i.$$

Thus H is a k -linear functor whose matrix of ranks is given by

$$r_{j,(\chi,g)} = \begin{cases} d_i, & \text{if } (j, \chi, g) \in X_{i,\beta_i} \text{ for some } i \in \{1, \dots, n\} \\ 0, & \text{otherwise} \end{cases}.$$

Now, the action of $\pi_0(\mathbb{G})$ on $X(n, pq) \cong [n] \times \pi_1(\mathbb{G})^* \times \pi_0(\mathbb{G})$ is given by

$$(i, \chi, g) \cdot \tilde{g} = (\rho(\tilde{g}^{-1})(i), \chi, g\tilde{g}^{-1})$$

(see § 5.2). Since $X_{i,\beta_i} = (i, \beta_i, e)\pi_0(\mathbb{G})$ it follows that

$$\begin{aligned} r_{j,(\chi,g)} \neq 0 &\Leftrightarrow \exists i \in \{1, \dots, n\}, \exists \tilde{g} \in \pi_0(\mathbb{G}) : (j, \chi, g) = (\rho(\tilde{g}^{-1})(i), \beta_i, \tilde{g}^{-1}) \\ &\Leftrightarrow \exists i \in \{1, \dots, n\} : j = \rho(g^{-1})(i), \chi = \beta_i, \end{aligned}$$

in which case we have $r_{j,(\chi,g)} = d_i = d_{\rho(g)(j)}$. If we define

$$J(\chi, g) := \{j \in \{1, \dots, n\} \mid \exists i \in \{1, \dots, n\} : \rho(g^{-1})(i) = j, \beta_i = \chi\}$$

it follows that

$$H(\eta_{\chi,g}) \cong \bigoplus_{j \in J(\chi,g)} d_{\rho(g)(j)} V_j.$$

In particular we have

$$(7.4) \quad H(\eta_{\chi,e}) \cong \bigoplus_{j \in J(\chi,e)} d_j V_j = \bigoplus_{\beta_j = \chi} d_j V_j,$$

where the condition $\chi = \beta_j$ in the last direct sum means that it is taken over all $j \in \{1, \dots, n\}$ such that $\beta_j = \chi$. Using now that H is k -linear we obtain that

$$(7.5) \quad H(\eta_U) \cong \bigoplus_{\chi \in \pi_1(\mathbb{G})^*} H(\eta_{\chi,e}) \cong \bigoplus_{\chi \in \pi_1(\mathbb{G})^*} \left(\bigoplus_{\beta_j = \chi} d_j V_j \right) \cong \bigoplus_{j=1}^n d_j V_j \cong V.$$

This proves (i).

To prove (ii) let us first remark that for any intertwiners $(H, \Phi), (\tilde{H}, \tilde{\Phi})$ from \mathbf{R} to any other representation \mathbf{F} we have a bijection

$$A : 2\text{Hom}_{\mathbb{G}}((H, \Phi), (\tilde{H}, \tilde{\Phi})) \xrightarrow{\cong} \prod_{i=1}^n \text{Hom}_k(k^{d_i}, k^{\tilde{d}_i})$$

between the set of 2-intertwiners $\tau : (H, \Phi) \Rightarrow (\tilde{H}, \tilde{\Phi})$, on the one hand, and the set of n -tuples of linear maps (f_1, \dots, f_n) with $f_i : k^{d_i} \rightarrow k^{\tilde{d}_i}$, on the other. It basically give the morphisms of vector bundles over the various intertwining orbits X_{i,β_i} , and its existence follows from (6.17). Moreover, it follows from Remark 26 that this bijection maps a 2-intertwiner τ to the n linear maps $A(\tau)_1, \dots, A(\tau)_n$ obtained in the following way. From (7.4) we know that

$$H(\eta_{\beta_i,e}) \cong \bigoplus_{j=1}^n r_{j,(\beta_i,e)} V_j = \bigoplus_{\beta_j = \beta_i} d_j V_j,$$

where the last direct sum is taken over all $j \in \{1, \dots, n\}$ such that $\beta_j = \beta_i$, and similarly for $\tilde{H}(\eta_{\beta_i, e})$. Hence we have

$$\tau_{\eta_{\beta_i, e}} : \bigoplus_{\beta_j = \beta_i} d_j V_j \rightarrow \bigoplus_{\beta_j = \beta_i} \tilde{d}_j V_j,$$

Then $A(\tau)_i : k^{d_i} \rightarrow k^{\tilde{d}_i}$ is the linear map whose matrix in canonical bases is equal to the matrix giving the restriction of $\tau_{\eta_{\beta_i, e}}$ to the V_i -‘isotypic’ component.

Suppose now we are given a morphism $\phi : H(\eta_U) \rightarrow \tilde{H}(\eta_U)$ in \mathcal{V} . Because of (7.5) and the absolute simplicity of the objects V_i , we see that giving such a morphism amounts to giving n arbitrary linear maps $f_i : k^{d_i} \rightarrow k^{\tilde{d}_i}$. Therefore we also have a bijection

$$B : \prod_{i=1}^n \text{Hom}_k(k^{d_i}, k^{\tilde{d}_i}) \xrightarrow{\cong} \text{Hom}_{\mathcal{V}}(H(\eta_U), \tilde{H}(\eta_U)),$$

mapping an n -tuple (f_1, \dots, f_n) to the unique morphism $\phi : H(\eta_U) \rightarrow \tilde{H}(\eta_U)$ whose restriction to the V_i -‘isotypic’ component is given by the matrix of f_i in canonical bases. It then follows that the composite bijection

$$B \circ A : 2\text{Hom}_{\mathbb{G}}((H, \Phi), (\tilde{H}, \tilde{\Phi})) \longrightarrow \text{Hom}_{\mathcal{V}}(H(\eta_U), \tilde{H}(\eta_U))$$

maps any τ to its component τ_{η_U} , and this proves (ii). \square

Corollary 31. *Under the same assumptions on \mathbb{G} and k as before we have an equivalence of k -additive categories*

$$\mathcal{E}nd(\omega) \simeq \text{Vect}_k^{\mathcal{G}}.$$

In particular, $\mathcal{E}nd(\omega)$ is a 2-vector space of rank pq .

Proof. Any equivalence $f : X \xrightarrow{\sim} Y$ in a $(k\text{-linear})$ 2-category \mathbf{C} induces equivalences of $(k\text{-linear})$ categories

$$\mathcal{E}nd_{\mathbf{C}}(X) \simeq \text{Hom}_{\mathbf{C}}(X, Y) \simeq \mathcal{E}nd_{\mathbf{C}}(Y).$$

In our case \mathbf{C} is the 2-category $\mathbf{PsFun}(\mathbf{Rep}_{2\mathbf{Vect}_k}(\mathbb{G}), 2\mathbf{Vect}_k)$, X is the hom-pseudofunctor $\text{Hom}_{\mathbb{G}}(\mathbf{R}, -)$, Y is ω and f is any representation of ω by \mathbf{R} . Hence we have

$$\mathcal{E}nd(\omega) \simeq \mathcal{P}Nat(\text{Hom}_{\mathbb{G}}(\mathbf{R}, -), \omega) \simeq \text{Vect}_k^{\mathcal{G}}$$

because of Yoneda. \square

Remark 32. It is worth comparing this with the situation we have for finite groups. Thus for any finite group G there also exists a “universal function” $f_U : G \rightarrow k$, i.e. a function such that for any representation V and any $v \in V$ there exists a unique morphism of representations $h : L(G) \rightarrow V$ with $h(f_U) = v$. Such a universal function is the function δ_e equal to zero everywhere except on the unit element $e \in G$ where it is equal to 1. Hence the analog in our categorified setting of the basic function δ_e is none of the basic functors $\eta_{\chi, e}$, for some particular $\chi \in \pi_1(\mathbb{G})^*$, but the direct sum of all of them.

7.4. Basis of $\mathcal{E}nd(\omega)$. From Theorem 30 and the above description of Yon^* (see (7.1)-(7.2)) it follows that a specific representation of $\omega : \mathbf{Rep}_{2\mathbf{Vect}_k}(\mathbb{G}) \rightarrow 2\mathbf{Vect}_k$ is the 2-natural equivalence

$$\Theta \equiv \text{Yon}^*(\eta_U) : \text{Hom}_{\mathbb{G}}(\mathbf{R}, -) \Rightarrow \omega$$

whose 1-cell components are the k -linear functors $\Theta_{\mathbf{F}} : \text{Hom}_{\mathbb{G}}(\mathbf{R}, \mathbf{F}) \rightarrow \mathcal{V}$ given on objects (H, Φ) and morphisms $\tau : H \Rightarrow H'$ by

$$\begin{aligned} \Theta_{\mathbf{F}}(H, \Phi) &:= H(\oplus_{\chi} \eta_{\chi, e}) \\ \Theta_{\mathbf{F}}(\tau) &:= \tau_{\oplus_{\chi} \eta_{\chi, e}}. \end{aligned}$$

This induces a $(k\text{-linear})$ equivalence

$$\mathcal{E}nd(\omega) \rightarrow \mathcal{P}Nat(\text{Hom}_{\mathbb{G}}(\mathbf{R}, -), \omega)$$

defined by $u \mapsto u \cdot \Theta$. Composing it with the Yoneda equivalence gives the desired k -linear equivalence $E : \mathcal{E}nd(\omega) \xrightarrow{\cong} \mathcal{V}ect_k^{\mathcal{G}}$. This turns out to be the equivalence mapping the pseudonatural transformation $u : \omega \Rightarrow \omega$ to the functor

$$u_{\mathbf{R}}(\oplus_{\chi} \eta_{\chi,e}) : \mathcal{G} \rightarrow \mathcal{V}ect_k,$$

and a modification $\mathbf{n} : u \Rightarrow u'$ to the natural transformation

$$\mathbf{n}_{\mathbf{R}, \oplus_{\chi} \eta_{\chi,e}} : u_{\mathbf{R}}(\oplus_{\chi} \eta_{\chi,e}) \Rightarrow u'_{\mathbf{R}}(\oplus_{\chi} \eta_{\chi,e}).$$

A pseudoinverse $E^* : \mathcal{V}ect_k^{\mathcal{G}} \rightarrow \mathcal{E}nd(\omega)$ is given by

$$E^*(\eta) = \text{Yon}^*(\eta) \cdot \Theta^*, \quad \eta \in \text{Obj} \mathcal{V}ect_k^{\mathcal{G}}$$

for some pseudoinverse Θ^* of the above 2-natural equivalence Θ . The 1-cell components of such a Θ^* are described in Remark 26 for \mathcal{V} of the form $\mathcal{V}ect_k^n$. For an arbitrary 2-vector space \mathcal{V} we just need to identify the standard basis of $\mathcal{V}ect_k^n$ with a basis of \mathcal{V} . In particular, for any representation $\mathbf{F} = (\mathcal{V}, \mathbb{F})$ the k -linear equivalence

$$\Theta_{\mathbf{F}}^* : \mathcal{V} \rightarrow \mathcal{H}om_{\mathbb{G}}(\mathbf{R}, \mathbf{F})$$

maps a basis $\{V_1, \dots, V_n\}$ of \mathcal{V} to the basic intertwiners whose matrices of ranks are given by (6.15). Hence we have the following.

Proposition 33. *A basis of $\mathcal{E}nd(\omega)$ as a 2-vector space is given by a family of endomorphisms*

$$\{\zeta_{\chi,g} \equiv E^*(\eta_{\chi,g}) : \omega \Rightarrow \omega, \ (\chi, g) \in \pi_1(\mathbb{G})^* \times \pi_0(\mathbb{G})\}$$

whose 1-cell components are given by

$$\zeta_{\chi,g;\mathbf{F}}(V_i) = \begin{cases} V_{\rho(g^{-1})(i)}, & \text{if } \chi = \beta_i, \\ 0, & \text{otherwise} \end{cases}$$

if $\mathbf{F} \simeq \mathbf{F}(n, \rho, \beta, c)$. In particular, $\zeta_{\chi,g}$ is totally supported on representations whose $\beta = (\beta_1, \dots, \beta_n)$ includes the character χ .

8. FINAL REMARK

A well known important property of the regular representation $L(G)$ of a finite group G is that it is equivalent (for algebraically closed fields k) to the direct sum of all nonequivalent irreducible representations, each one with a multiplicity exactly equal to its own dimension. Because of the similarities we have found until now one might be tempted to think that the same is true for essentially finite 2-groups. However, on the one hand, in our setting there may exist non-irreducible but indecomposable representations. This fact has been pointed out in [10] in the even more general framework of representations of 2-groups in Yetter's measurable categories, of which our representation theory is a special case. Hence not every representation will necessarily decompose as a direct sum of irreducible ones. On the other hand, as pointed out before, there is no Schur's Lemma in our representation theory, at least in its usual form, and such lemma seems to be crucial to prove the above mentioned result for finite groups. Indeed, if G is a finite group and k an algebraically closed field Schur's lemma implies that for any irreducible representation V_i of G the dimension of $\text{Hom}_G(L(G), V_i)$ is precisely equal to the multiplicity of V_i in $L(G)$. The result mentioned above follows then because ω is represented by $L(G)$ so that we have $V_i \cong \text{Hom}_G(L(G), V_i)$. When we move to our setting, we still have an equivalence of 2-vector spaces $\mathcal{V} \simeq \mathcal{H}om_{\mathbb{G}}(\mathcal{V}ect_k^{\mathcal{G}}, \mathcal{V})$ for any linear representation $(\mathcal{V}, \mathbb{F})$ of \mathbb{G} . However, it is not at all clear whether the rank of $\mathcal{H}om_{\mathbb{G}}(\mathcal{V}ect_k^{\mathcal{G}}, \mathcal{V})$, for $(\mathcal{V}, \mathbb{F})$ irreducible, coincides with the 'number of copies' of it in the regular representation.

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DEPT. MATEMÀTICA APLICADA II, UNIVERSITAT POLITÈCNICA DE CATALUNYA
 E-mail address: josep.elgueta@upc.edu